# The Pure Virtual Braid Group is Quadratic

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**Abstract**

If an augmented algebra $K$ over $\mathbb{Q}$ is filtered by powers of its augmentation ideal $I$, the associated graded algebra $gr_I K$ need not in general be quadratic: although it is generated in degree 1, its relations may not be generated by homogeneous relations of degree 2. In this paper we give a criterion which is equivalent to $gr_I K$ being quadratic. We apply this criterion to the group algebra of the pure virtual braid group (also known as the quasi-triangular group), and show that the corresponding associated graded algebra is quadratic.

## Contents

# 1 Introduction

This paper will ultimately be concerned with the pure virtual braid groups $\boldsymbol{PvB_n}$, for all $n \in \mathbb{N}$, generated by symbols $R_{ij}$, $1 \leq i \neq j \leq n$, with relations the Reidemeister III moves (or quantum Yang-Baxter relations) and certain commutativities:

$$R_{ij}R_{ik}R_{jk} = R_{jk}R_{ik}R_{ij} \tag{1}$$
$$R_{ij}R_{kl} = R_{kl}R_{ij}, \tag{2}$$

with $i, j, k, l$ distinct. This group is referred to as the quasi-triangular group $\boldsymbol{QTr_n}$ in [BarEnEtRa]. We will also be concerned with the related algebra $\mathfrak{pvb}_n$, generated by symbols $r_{ij}$, $1 \leq i \neq j \leq n$, with relations the '6-term' (or 'classical Yang-Baxter') relations, and related commutativities:

$$y_{ijk} := [r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] = 0, \tag{3}$$
$$c_{ij}^{kl} := [r_{ij}, r_{kl}] = 0 \tag{4}$$

with $i, j, k, l$ distinct. This algebra is the universal enveloping algebra of the quasi-triangular Lie algebra $\mathfrak{qtr_n}$ in [BarEnEtRa].

We will show that $PvB_n$ is a 'quadratic group', in the sense that if its rational group ring $\mathbb{Q}PvB_n$ is filtered by powers of the augmentation ideal $I$, the associated graded ring $grPvB_n$ is a quadratic algebra: i.e., a graded algebra generated in degree 1, with relations generated by homogeneous relations of degree 2. We note that, in different language, this is the statement that $PvB_n$ has a universal finite-type invariant, which takes values in the algebra $\mathfrak{pvb}_n$.

In [Hutchings], a criterion was given for the quadraticity of the pure braid group. The proof relied on the geometry of braids embedded in $\mathbf{R}^3$. In order to generalize this criterion to all finitely presented groups, we developed an algebraic proof of the criterion. This proof turned out not to rely on the existence of an underlying group, and applies instead to algebras over $\mathbb{Q}$, filtered by powers of an augmentation ideal $I$. Indeed, this criterion arguably lives naturally in an even broader context, such as perhaps augmented algebras over an operad (or the related 'circuit algebras' of [BN-WKO]), although we do not investigate this broader context here.

Our criterion may be summarized as follows in the case of an augmented algebra $K$ with augmentation ideal $I_K$. We denote by $gr_I K = \oplus_{m \geq 0} I_K^m / I_K^{m+1}$ the associated graded algebra of $K$ with respect to the filtration by powers of the augmentation ideal. Let $A$ be the 'quadratic approximation' of $gr_I K$, namely the graded algebra with the same generators and with ideal of relations generated by the degree 2 relations of $gr_I K$. We will see that, in fact, we can view the generators of $K$ as also generating $A$, and interpret a certain space $\mathfrak{R}^F$ of free generators of the relations in $K$ as generating the relations in $A$. It thus makes sense to ask whether the relations among the elements of $\mathfrak{R}^F$, when viewed as relations in $A$, also hold when these are viewed as relations in $K$ - i.e., informally, whether the syzygies in $A$ also hold in $K$. We show that $gr_I K$ is quadratic if and only if the syzygies of $A$ do also hold in $K$. Furthermore, if $A$ is Koszul, we show that it is sufficient to check this criterion in degrees 2 and 3.

In Section 1 of this paper we set the stage for and give a precise statement of this criterion (see Theorem 1). In Section 2 we supply details of proofs that were omitted in Section 1. In Section 3 we specialize to $PvB_n$. We present a basis for the quadratic dual algebra $\mathfrak{pvb}_n^!$, and use this basis to compute the syzygies of $\mathfrak{pvb}_n$ and prove that $PvB_n$ satisfies the quadraticity criterion. It follows that $PvB_n$ is quadratic.

Although Koszulness of the algebra $\mathfrak{pvb}_n$ was originally established in [BarEnEtRa], we give a different proof in Subsection 3.7, by exhibiting a quadratic Gröbner basis for $\mathfrak{pvb}_n^!$. In Subsection 3.8, we further use this quadratic Gröbner basis to prove that $\mathfrak{pvb}_n$ is a free $\mathfrak{pvb}_{n-1}$ module with respect to the module structure induced by the inclusion $\mathfrak{pvb}_{n-1} \hookrightarrow \mathfrak{pvb}_n$, for all $n \geq 3$.

Finally, in Section 4 we point out some possible future avenues of research.

We note that the quadraticity of $PvB_n$ was conjectured in [BarEnEtRa]. As pointed out in section 8.5 of that paper, the quadraticity of $PvB_n$ implies that $H^*(PvB_n) \cong \mathfrak{pvb}_n^!$ as algebras.

After this paper was substantially completed, the result was communicated to Alexander Polishchuk, who pointed out that a theorem similar to Theorem 1 was obtained in [PosVish] in the context of the cohomology algebra of a nilpotent augmented coalgebra, albeit by different methods. For this reason we have referred to the criterion in Theorem 1 as the PVH Criterion (with reference to Positselski, Vishik and Hutchings).

[NTD: Inclusion of the following remark is still tentative:

**Remark 1.** *Rational Homotopy Theory Interpretation*

Recall that a group $G$ is referred to as 1-formal if the Malcev Lie algebra $M_G$ of $G$ is isomorphic to the (completed) rational holonomy Lie algebra $hol(G)$ of $G$. One can also consider a weaker, graded version of this concept, i.e. we consider a group to be 'graded 1-formal' if $grM_G \cong gr(hol(G))$, with the associated gradeds being determined with respect to the lower central series filtrations. This corresponds to the associated graded of the universal enveloping algebra $U(M_G)$ of $M_G$ (with respect to the filtration by powers of the augmentation ideal) being a quadratic algebra. In this formulation, the PVH Criterion gives a criterion for a group being graded 1-formal.

NTD: I will include this remark if I can establish that $q(grU(M_G)) = U(holg(G))$ for all (reasonable) $G$.]

## 1.1 Overview of the PVH Criterion

### 1.1.1 Group Theoretic Background

Since the classic setting of the PVH criterion is that of group rings, we identify the attributes of group rings which we rely on and will want to see preserved in our generalized context. We recall the follow basic fact:

**Proposition 1** (See [MKS], s. 5.15)**.** *If $G$ is given by the short exact sequence*

$$1 \to N \to FG \to G \to 1$$

*where $FG$ is a free group generated by symbols $\{g_p : p \in P\}$ and $N$ is a normal subgroup of $FG$ generated by the set $\{r_q : q \in Q\}$, then the rational group ring of $G$ is given by the exact sequence*

$$0 \to (N-1) \to \mathbb{Q}FG \to \mathbb{Q}G \to 0$$

*where $(N-1)$ is the two-sided ideal in $\mathbb{Q}FG$ generated by $\{(r_q - 1) : q \in Q\}$.*

We can clearly restrict the second exact sequence to the exact sequence

$$0 \to (N-1) \to I_{FG} \to I_G \to 0 \tag{5}$$

where $I_{FG}$ and $I_G$ are the augmentation ideals of $\mathbb{Q}FG$ and $\mathbb{Q}G$ respectively.

### 1.1.2 Generalized Algebraic Setting

By analogy with the above group case, we take $\boldsymbol{K}$ to be an augmented (unital) algebra over $\mathbb{Q}$ with 2-sided augmentation ideal $I_K$, and $\boldsymbol{F}$ to be the free algebra over $\mathbb{Q}$ with the same generating set as $K$, with 2-sided augmentation ideal $\boldsymbol{I_F}$. In particular we assume an exact sequence:

$$0 \longrightarrow I_K \longrightarrow K \xrightarrow{\epsilon} \mathbb{Q} \longrightarrow 0$$

By analogy with the ideal $(N-1)$ in the group context, we let $\boldsymbol{M} \subseteq I_F \subseteq F$ be a 2-sided ideal such that:

$$0 \longrightarrow M \longrightarrow F \longrightarrow K \longrightarrow 0$$
$$0 \longrightarrow M \longrightarrow I_F \longrightarrow I_K \longrightarrow 0$$

are exact.

We suppose $F$ (and $K$) to be generated by a set $\boldsymbol{X}$. For convenience, we will suppose $I_F$ (and $I_K$) to be the kernel of the algebra homomorphism which sends each $x \in X$ to $1 \in \mathbb{Q}$. Using this convention, we will exhibit an explicit grading on $F$ which induces the filtration by powers of $I_F$.

Specifically, we define $\tilde{\boldsymbol{X}} := \mathbb{Q}\{\bar{x} := (x-1) : x \in X\}$. Then $F$ has the graded structure of tensor algebra[1] over $\tilde{X}$, i.e. $F = T\tilde{X} = \oplus_{p \geq 0} \tilde{X}^p$ where $\tilde{\boldsymbol{X}}^{\boldsymbol{p}}$ consists of all sums of $p$-fold products of elements of $\tilde{X}$. We obtain a filtration on $F$ by setting $\tilde{\boldsymbol{X}}^{\geq \boldsymbol{p}} := \oplus_{q \geq p} \tilde{X}^q$. It should be clear that $\tilde{X}^{\geq p} = I_F^p$.

We will henceforth in fact work with the completions $\hat{K}$ of $K$ (and $\hat{F}$ of $F$) with respect to the filtrations by powers of their respective augmentation ideals. Our reason for doing this is that, by picking a suitable set of generators for $K$ (and $F$) and passing to the completions, we claim that we may arrange that $M \subseteq I_F^2$ (see Subsection 2.1), and we will need this in the sequel. Since we always work with the completions, we will simply denote them $K$ and $F$, without the hat.

### 1.1.3 The Associated Graded Algebra

$K$ is filtered by powers of $I_K$:

$$\ldots \hookrightarrow I_K^3 \hookrightarrow I_K^2 \hookrightarrow I_K \hookrightarrow I_K^0 = K$$

We denote $\boldsymbol{gr_I K}$ the associated graded of the above filtration. We have $gr_I K \cong \bigoplus_p I_K^p / I_K^{p+1}$. It is clear that $gr_I K$ is generated as an algebra by its degree one piece $\boldsymbol{V} := I_K / I_K^2$, a vector space over $\mathbb{Q}$.

### 1.1.4 A Candidate Presentation for $I_K^p$

In order to understand the quotients $I_K^p / I_K^{p+1}$, we will first seek a presentation for the $I_K^p$, for $p \geq 0$ (we take $I_K^0 = K$). Note that, essentially by definition, $I_K^p = I_F^p / (M \cap I_F^p)$. So we wish to determine $(M \cap I_F^p)$.

Let $\{y_q : q \in Q\} \subseteq I_F^2$ be a minimal set of generators for $M$. Then define $\boldsymbol{\mathcal{Y}} := \{Y_q : q \in Q\}$ to be a collection of symbols in 1-1 correspondence with the $\{y_q\}$. Then we define $\boldsymbol{\mathfrak{R}^F}$ to be the free 2-sided $F$-module generated by $\mathcal{Y}$. Now if we define a map $\boldsymbol{\partial_K} : \mathfrak{R}^F \to F$ which maps $Y_q \mapsto y_q$ (and extend $\partial_K$ as an $F$-module homomorphism to $\mathfrak{R}^F$) then it is clear that

[1] This can be seen as follows. Essentially by definition, $F = TX$, the tensor algebra over $X$. But then it is easy to see that the algebra homomorphism which maps $x \mapsto \bar{x} + 1$ is actually an automorphism of $F$ which converts from the $TX$ presentation to the $T\tilde{X}$ presentation.

$$K = \frac{F}{\partial_K \mathfrak{R}^F} \tag{6}$$

$\mathfrak{R}^F$ inherits both a graded and a filtered structure from $F$. Specifically, if we define:

$$\mathfrak{R}_{p,q} := \tilde{X}^q.\mathbb{Q}\mathcal{Y}.\tilde{X}^{p-q-2}$$

and

$$\mathfrak{R}_p := \sum_{q=0}^{p-2} \mathfrak{R}_{p,q} \tag{7}$$

then $\mathfrak{R}^F = \oplus_{p\geq 0}\mathfrak{R}_p$ gives a graded structure on $\mathfrak{R}^F$. We get a filtered structure by defining

$$\boldsymbol{\mathfrak{R}_{\geq p}} := \bigoplus_{q\geq p} \mathfrak{R}_q$$

and we have the filtration of $\mathfrak{R}^F$:

$$\ldots \hookrightarrow \mathfrak{R}_{\geq 3} \hookrightarrow \mathfrak{R}_{\geq 2} = \mathfrak{R}^F$$

By construction, $\partial_K(\mathfrak{R}_{\geq p}) \subseteq (M \cap I_F^p)$. In fact we will prove in Section 2.2 that we have the following:

**Proposition 2.** *If the Hutchings Criterion (to be defined in Theorem 1) is met, then for all $p \geq 2$ we have $\partial_K(\mathfrak{R}_{\geq p}) = (M \cap I_F^p)$, and hence $I_K^p \cong I_F^p/\partial_K \mathfrak{R}_{\geq p}$.*

It is clear that $\partial_K$ is a filtered map, ie $\partial_K : \mathfrak{R}_{\geq p} \to I_F^p = \tilde{X}^{\geq p}$. This leads to the following remark.

**Remark 2.** *Spectral Sequence Interpretation*

Although not required logically to prove the PVH Criterion (nor even to understand it), it may be of interest to note that the PVH Criterion can be explained in terms of a spectral sequence.

Specifically, we can interpret the construction of $K$ in (6) in homological terms. Indeed, we can view the map $\partial_K : \mathfrak{R}^F \to F$ as forming a complex:

$$0 \to C_1 = \mathfrak{R}^F \xrightarrow{\partial_K} C_0 = F \xrightarrow{0} 0$$

Then $K = H_0(C_\bullet)$. Since $\partial_K$ respects the filtrations on $\mathfrak{R}^F$ and $F$, we can compute $grH_0$ (the associated graded of $H_0$ with respect to the induced filtration on $H_0$) using a spectral sequence. Moreover, under this interpretation, $grH_0 = gr_I K$, and (as we will see in the next two subsections) the terms $E_1^{p,-p}$ of the first page correspond to the grade-$p$ components of the quadratic approximation $q(gr_I K)$ to $gr_I K$. Theorem 1 can be interpreted as saying that if the Hutchings Criterion (defined in that theorem) is met, then $E_1^{p,-p} = E_\infty^{p,-p}$ and $q(gr_I K) \cong \oplus E_1^{p,-p} \cong grH_0 \cong gr_I K$ as vector spaces.

We will periodically provide further details of this interpretation as they become relevant.

### 1.1.5 A Graded Approximation to $K$

We may 'approximate' $K$ by taking a graded version of the construction (6) of $K$, i.e. we define an algebra

$$\boldsymbol{A} := \frac{\oplus_{p\geq 0}\tilde{X}^p}{\oplus_{p\geq 0}\partial_A\mathfrak{R}_p} \tag{8}$$

where $\partial_A$ is a graded version of $\partial_K$. More precisely, we have projections:

$$\pi_p^0 : \tilde{X}^{\geq p} \twoheadrightarrow \tilde{X}^p$$

which are the identity on $\tilde{X}^p$ and send $\oplus_{q>p}\tilde{X}^q$ to 0. Similarly, we have projections:

$$\pi_p^1 : \mathfrak{R}_{\geq p} \twoheadrightarrow \mathfrak{R}_p$$

which are the identity on $\mathfrak{R}_p$ and send $\oplus_{q>p}\mathfrak{R}_q$ to 0.

Then $\partial_A$ is defined by:

$$\partial_A(z) = \pi_p^0 \circ \partial_K(z) \text{ for } z \in \mathfrak{R}_p$$

These definitions are encapsulated in the following diagram, which is commutative and has exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{R}_{\geq p+1} & \xrightarrow{\iota} & \mathfrak{R}_{\geq p} & \xrightarrow{\pi_p^1} & \mathfrak{R}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_K^{res}} & & \downarrow{\scriptstyle \partial_K} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \longrightarrow & I_F^{p+1} & \xrightarrow{\iota} & I_F^p & \xrightarrow{\pi_p^0} & \tilde{X}^p & \longrightarrow & 0
\end{array}$$

where $\partial_K^{res}$ denotes the restriction and $\iota$ denotes the inclusions.

**Remark 3.** *Spectral Sequence Interpretation Continued*

We explain briefly how the algebra $A$ fits with the spectral sequence interpretation from Remark 2. Essentially by definition, we have:

$$E_0^{p,-p} = \tilde{X}^p$$
$$E_1^{p,1-p} = \mathfrak{R}_p$$

and $E_\bullet^{p,q} = 0$ for $q \neq -p, 1-p$. Furthermore, the page 0 differential $d_0$ for the spectral sequence satisfies $d_0 = \partial_A$, so $E_1^{p,-p} = A^p$, and $E_1^{p,1-p} = [ker\ \partial_A]_p$ (the degree $p$ component of $ker\ \partial_A$).

As we will see shortly, the PVH Criterion will imply that $d_1$ (and any higher degree differentials) do not add any (non-zero) corrections to the $E_1$ page.

### 1.1.6 The Quadratic Approximation

In this subsection, we show that the graded algebra $A$ is the 'quadratic approximation' $q(gr_I K)$ of $gr_I K$, in the sense that $A$ has the same generators as $gr_I K$, and has relations generated by the degree 2 relations of $gr_I K$.

We noted previously that $gr_I K$ is generated by its degree 1 piece $V = I_K/I_K^2$. In fact, because we take $M \subseteq I_F^2$, we have

$$I_K/I_K^2 = \frac{I_F/M}{I_F^2/(M \cap I_F^2)} \cong \frac{I_F/M}{I_F^2/M} \cong I_F/I_F^2$$

so that actually we may view $V$ as $V = I_F/I_F^2$.

With that in mind, it should be clear that we have an isomorphism

$$\tilde{X} \xrightarrow{\sim} V = I_F/I_F^2$$
$$(x-1) \mapsto (x-1) + I_F^2$$

which extends to an isomorphism of graded algebras $\Pi : F = T\tilde{X} \xrightarrow{\sim} TV$ (where $TV$ is the tensor algebra of $V$ over $\mathbb{Q}$) by the universal property of tensor algebras.

Now we identify $V \otimes V = I_F/I_F^2 \otimes I_F/I_F^2$, and define $R := ker(m_K : I_F/I_F^2 \otimes I_F/I_F^2 \to I_K^2/I_K^3)$. Here $m_K$ is the composition $I_F/I_F^2 \otimes I_F/I_F^2 \overset{m_F}{\to} I_F^2/I_F^3 \overset{p}{\twoheadrightarrow} I_K^2/I_K^3$, where the first map $m_F$ is the isomorphism induced from multiplication in $F$, and the second map $p$ is induced from the projection $F \twoheadrightarrow K$.

The quadratic approximation $q(gr_I K)$ is formally defined as $q(gr_I K) := TV/\langle R \rangle$, where $\langle R \rangle$ is the two-sided ideal in $TV$ generated by the vector subspace $R \subseteq V \otimes V$. Thus $A$ and $q(gr_I K)$ at least have spaces of generators which are isomorphic via the map $\Pi$. The following lemma effectively tells us that $\partial_A \mathbb{Q}\mathcal{Y} = R$ and hence that $A$ and $q(gr_I K)$ have the same relations:

**Lemma 1.** *We have* $R \cong (M + I_F^3)/I_F^3 \cong \partial_A \mathbb{Q}\mathcal{Y}$.

*Proof.* It suffices to determine the kernel $ker(I_F^2/I_F^3 \overset{p}{\twoheadrightarrow} I_K^2/I_K^3)$, and then pull the result back to $V \otimes V$ via the isomorphism $m_F^{-1}$. We have:

$$\frac{I_K^2}{I_K^3} = \frac{I_F^2/M}{I_F^3/(M \cap I_F^3)} \cong \frac{I_F^2/M}{(I_F^3 + M)/M} \cong \frac{I_F^2}{I_F^3 + M} \cong \frac{I_F^2/I_F^3}{(I_F^3 + M)/I_F^3}$$

It follows that $ker\ p$ must be the last denominator, i.e. $(I_F^3 + M)/I_F^3$, which is what we needed. ☐

It is immediate from the lemma that in fact $\oplus_{p\geq 2}\partial_A \mathfrak{R}_p = \langle R \rangle$, and hence that $A \cong q(gr_I K)$.

We will denote by $A^m$ the $m$-th graded piece of $A$. We note that since $A$ has the same generators and the same quadratic relations as $gr_I K$, there is always a surjection $A \twoheadrightarrow gr_I K$. Quadraticity of $gr_I K$ is thus equivalent to the fact that this surjection is an isomorphism $A^m \cong I_K^m/I_K^{m+1}$, for all $m$. We will often use this alternative definition of quadraticity.

### 1.1.7 The Hutchings Criterion

We now resume the thread of our development of the Hutchings Criterion. Recall that we have the exact, commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{R}_{\geq p+1} & \longrightarrow & \mathfrak{R}_{\geq p} & \xrightarrow{\pi_p^1} & \mathfrak{R}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_K^{res}} & & \downarrow{\scriptstyle \partial_K} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \longrightarrow & I_F^{p+1} & \longrightarrow & I_F^p & \xrightarrow{\pi_p^0} & \tilde{X}^p & \longrightarrow & 0
\end{array}$$

We extend the right half of the above diagram by adding kernels at the top:

$$\begin{array}{ccc}
ker\ \partial_K & \xrightarrow{\pi^{Syz}} & ker\ \partial_A \\
\downarrow & & \downarrow \\
\mathfrak{R}_{\geq p} & \overset{\pi_p^1}{\twoheadrightarrow} & \mathfrak{R}_p \\
\downarrow{\scriptstyle \partial_K} & & \downarrow{\scriptstyle \partial_A} \\
I_F^p & \overset{\pi_p^0}{\twoheadrightarrow} & \tilde{X}^p
\end{array}$$

where $\pi^{Syz}$ is the map induced from $\pi_p^1$ on kernels; and also we have abbreviated $\partial_K|_{\mathfrak{R}_{\geq p}}$ as $\partial_K$, and $\partial_A|_{R_p}$ as $\partial_A$.

We are now in a position to state our criterion for quadraticity of $K$ (with notation as in the above diagram, and with the assumptions in Subsection 1.1.2)[2]:

**Theorem 1** (PVH Criterion)**.** *$K$ is quadratic if and only if $\pi^{Syz}$ is surjective for all $m \geq 2$, i.e. informally iff 'the syzygies of $A$ also hold in $K$'.*

*If $A$ is Koszul,[3] then we need only check that this criterion holds for degree 2 and 3 syzygies of $A$.*

This generalizes a result first obtained in [Hutchings], where $K$ was the group ring of the pure braid group (see also [BNStoi]). We give the proof in Section 2.2. As was pointed out to me by Alexander Polishchuk, the result also follows from the paper [PosVish], whenever the algebra $K$ is finitely generated.

### 1.1.8 Checking the Hutchings Criterion in Degree 2

The following proposition shows how to check the criterion in degree 2.

[2] The statement about Koszulness, however, relies on results about Koszul algebras which have only been developed for graded algebras whose graded components are finitely generated over the ground ring. Hence, for purposes of this part of the theorem, we assume the algebra $K$ to be finitely generated, which is sufficient to ensure that $A^m$ is a finite dimensional $\mathbb{Q}$-vector space for all $m$.

[3] In fact, $A$ need only be 2-Koszul, i.e. its Koszul complex need only be exact up to homological degree 2 inclusive.

**Proposition 3.** *Let $\{y_q : q \in Q\}$ be a minimal set of generators for $M$ as a two-sided $F$-module. If the $\{y_q + I_F^3 : q \in Q\}$ are linearly independent in $R \cong (M + I_F^3)/I_F^3$, then the PVH Criterion is satisfied in degree 2.*

*Proof.* Indeed, $\partial_A : \mathbb{Q}\mathcal{Y} \to V^{\otimes 2} \cong I_F^2/I_F^3$ is then an inclusion, so $ker\ \partial_A = 0$ and $\pi^{Syz}$ is automatically surjective. □

## 1.2 How the PVH Criterion is Useful

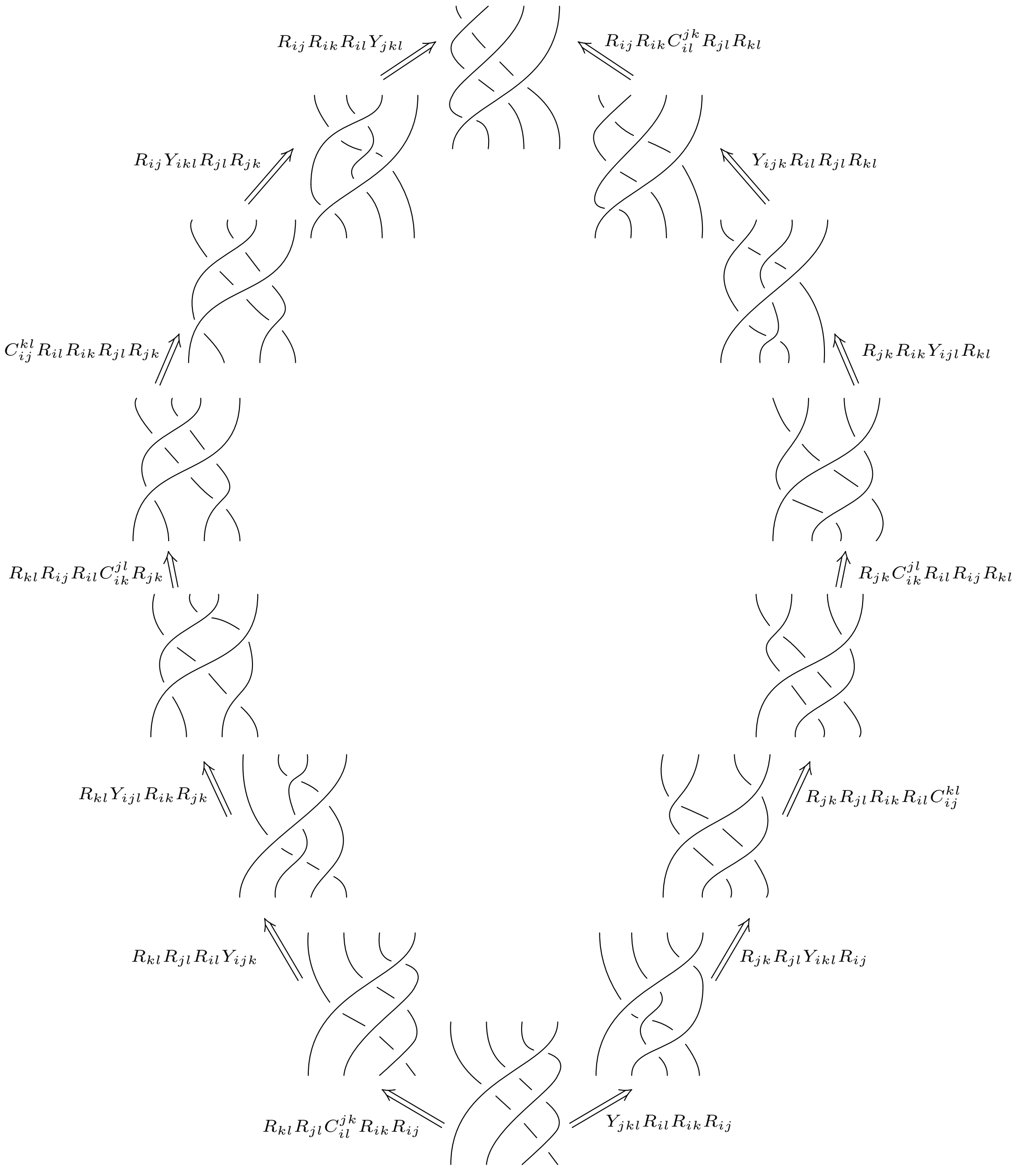
Assuming the requirements of Theorem 1 are met, we can conclude that $gr_I K$ is quadratic if, informally, the syzygies of $A$ also hold in $K$.

It is often the case that the syzygies of a quadratic algebra can be determined quite explicitly, using quadratic duality. Essentially, if the quadratic algebra $A$ is Koszul, then the syzygies are generated by $A^{!3}$ (i.e. the degree 3 part of the quadratic dual $A^!$ of $A$). Thus the problem of comparing syzygies is reduced to the finite, computable problem of determining a basis for $A^{!3}$ and checking whether the resulting syzygies of $A^3$ also hold in $K$.

In the context of $PvB_n$, it was shown in [BarEnEtRa] that $\mathfrak{pvb}_n$ is Koszul (a different proof is provided in Section 3.7 of this paper), so we only need to check the PVH Criterion in degree 2 and 3.

If we take $K$ to be the group ring of $PvB_n$ and $I_K$ its augmentation ideal, it is possible to interpret the ideals $I_K^m$ as spaces of '$m$-singular virtual braids' – essentially virtual braids with (at least) $m$ 'semi-virtual' double points (subject to a certain equivalence relation) - see [GPV]. One knows certain syzygies that are satisfied by such semi-virtual braids, particularly the syzygy known as the Zamolodchikov tetrahedron:[4]

[4]The picture builds on xy-pic templates due to Aaron Lauda – see [Lau]. Another picture is at [BN2].




(The notation will be clarified in Subsection 3.2.)

In the second part of this paper, we will find a basis for the quadratic dual algebra $\mathfrak{pvb}_n^!$, and in particular for $\mathfrak{pvb}_n^{!3}$. We will then check 'by hand' that the corresponding degree 3 syzygies of $A$ are also satisfied by $K$. These consist primarily of syzygies which correspond to the 'Zamolodchikov' syzygy alluded to above (this is explained in Subsection 3.4.1). This will allow us to conclude that $gr_I \mathbb{Q}PvB_n \cong \mathfrak{pvb}_n$.

### 1.3 Acknowledgements

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In addition I would like to thank Leonid Positselski for extremely useful and detailed comments concerning the interplay between the paper [PosVish] and this paper.

## 2 Postponed Proofs

### 2.1 Eliminating Linear Relations

We mentioned in Subsection 1.1.2 that we work with the completions $\hat{K}$ of $K$ (and $\hat{F}$ of $F$) because, by picking a suitable set of generators for $K$ (and $F$) and passing to the completions, we may arrange that $M \subseteq I_F^2$.

To prove this claim, we let $\{x_p : p \in P\}$ be a set of generators for the algebra $K$, so that $\{\bar{x}_p := (x_p - 1) : p \in P\}$ is a set of generators for $I_K$ as a left- or right-sided ideal in $K$. The images of the $\{\bar{x}_p : p \in P\}$ in the vector space $(I_K/I_K^2)$ generate that space, so the images of some subset $\{\bar{x}_p : p \in S \subseteq P\}$ form a basis. Thus the $\{\bar{x}_p : p \in P - S\}$ may be expressed as linear combinations of the $\{\bar{x}_p : p \in S\}$ modulo elements of $I_K^2$. More generally, we may replace any polynomial involving the $\{\bar{x}_p : p \in P - S\}$ by a polynomial involving only $\{\bar{x}_p : p \in S\}$, modulo elements in higher powers of $I_K$. It therefore follows that the $\{\bar{x}_p : p \in S\}$ generate the completion, and we may drop the $\{\bar{x}_p : p \in P - S\}$ from our list of generators.

We note in particular that in the case where $K$ is the group algebra of some group $G$, the generators of $K$ as an algebra would normally include, not only the group generators, but also their inverses. Moreover, the relations ideal $M$ would include relations derived from the group laws for the generators (recall that $F$ is the free algebra on $X$, not the free group algebra on $X$). Thus if $a$ is a generator of the group, and $b$ its inverse, we have the group law $ab = 1$ which gives, under the substitution $a \mapsto \bar{a} + 1$, $b \mapsto \bar{b} + 1$, where $\bar{a} := (a - 1)$ and $\bar{b} := (b - 1)$, the relation $\bar{a} + \bar{b} + \bar{a}\bar{b} = 0$, which is not in $I_F^2$. However using the relation $\bar{b} = -\bar{a} - \bar{a}\bar{b}$ we can replace all occurrences of $\bar{b}$ by $-\bar{a}$, provided we are working in the completion of $K$. So in the case of group algebras we will take as generators only the group generators and we omit the group law relations from $M$.

Coming back to the case of a general $K$, we can also see that $M \subseteq I_F^2$. Indeed, $(I_F/I_F^2)$ and $(I_K/I_K^2)$ are now vector spaces with bases having the same number of elements, and hence are isomorphic. However it is also clear that:

$$
\begin{aligned}
(I_K/I_K^2) &= \frac{I_F/M}{I_F^2/(M \cap I_F^2)} \\
&= \frac{I_F/M}{(I_F^2 + M)/M} \\
&= \frac{I_F}{I_F^2 + M} \\
&= \frac{(I_F/I_F^2)}{(I_F^2 + M)/I_F^2} \\
&= \frac{(I_F/I_F^2)}{M/(M \cap I_F^2)}
\end{aligned}
$$

so we must have $M/(M \cap I_F^2) = 0$, i.e. $M \subseteq I_F^2$.

## 2.2 Proof of Proposition 2 and Theorem 1

Recall that we have the following exact, commutative diagram:

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{R}_{\geq p+1} & \longrightarrow & \mathfrak{R}_{\geq p} & \xrightarrow{\pi_p^1} & \mathfrak{R}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_K^{res}} & & \downarrow{\scriptstyle \partial_K} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \longrightarrow & I_F^{p+1} & \longrightarrow & I_F^p & \xrightarrow{\pi_p^0} & \tilde{X}^p & \longrightarrow & 0
\end{array}
$$

where $\partial_K^{res}$ denotes the restriction.

We extend the above diagram by adding a row of kernels at the top, and a row of cokernels at the bottom:

$$
\begin{array}{ccccccccc}
 & & ker\ \partial_K^{res} & \longrightarrow & ker\ \partial_K & \xrightarrow{\pi^{Syz}} & ker\ \partial_A & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{R}_{\geq p+1} & \longrightarrow & \mathfrak{R}_{\geq p} & \xrightarrow{\pi_p^1} & \mathfrak{R}_p & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \partial_K} & & \downarrow{\scriptstyle \partial_K} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \longrightarrow & I_F^{p+1} & \longrightarrow & I_F^p & \xrightarrow{\pi_p^0} & \tilde{X}^p & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \frac{I_F^{p+1}}{\partial_K^{res}\mathfrak{R}_{\geq p+1}} & \xrightarrow{\mu} & \frac{I_F^p}{\partial_K \mathfrak{R}_{\geq p}} & \longrightarrow & A^p & &
\end{array}
$$

where we have abbreviated $\partial_K|_{\mathfrak{R}_{\geq p}}$ as $\partial_K$, and $\partial_A|_{R_p}$ as $\partial_A$.

By the Snake Lemma, the following sequence is exact:

**Lemma 2.**

$$0 \to ker\ \partial_K \xrightarrow{\pi^{Syz}} ker\ \partial_A \to \frac{I_F^{p+1}}{\partial_K \mathfrak{R}_{\geq p+1}} \xrightarrow{\mu} \frac{I_F^p}{\partial_K \mathfrak{R}_{\geq p}} \longrightarrow A^p \longrightarrow 0 \quad (9)$$

Also, as is clear from the long exact sequence, we have:

**Lemma 3.** *For every $p \geq 2$, $\pi^{Syz}$ is surjective if and only if $\mu : \frac{I_F^{p+1}}{\partial_K \mathfrak{R}_{\geq p+1}} \to \frac{I_F^p}{\partial_K \mathfrak{R}_{\geq p}}$ is injective.*

*Proof of Proposition 2.* It is clear that the $\mu : \frac{I_F^p}{\partial_K \mathfrak{R}_{\geq p}} \to \frac{I_F^{p-1}}{\partial_K \mathfrak{R}_{\geq p-1}}$ are injective for all $p \geq 2$ if and only if the compositions $\frac{I_F^p}{\partial_K \mathfrak{R}_{\geq p}} \to \frac{I_F^{p-1}}{\partial_K \mathfrak{R}_{\geq p-1}} \to \cdots \to I_K^p$ are also injective for all $p \geq 2$. Since these compositions are always surjective, injectivity is equivalent to isomorphism. This proves Proposition 2. □

*Proof of Theorem 1.* The first claim in Theorem 1 follows from the long exact sequence (9) and Proposition 2, which imply that

$$A^p \cong \frac{I_F^p}{\partial_K \mathfrak{R}_{\geq p}} / \frac{I_F^{p+1}}{\partial_K \mathfrak{R}_{\geq p+1}} \cong I_K^p / I_K^{p+1}$$

whenever $\pi^{Syz}$ is surjective.

We deal with the restriction to degrees 2 and 3 for the Koszul case in the next subsection. □

**Remark 4.** *Spectral Sequence Interpretation Continued*

In Remark 3 we interpreted the spaces $E_0$ and $E_1$ in terms of the algebra $A$ and its relations. The next step in applying a spectral sequence would be to successively determine the differentials $d_i$, $i \geq 1$, and thence the pages $E_{i+1}$, until we reach $E_\infty$. Instead, however, we can collapse all these steps into one.

The $d_1$ differential is normally determined[5] as the coboundary operator arising from the exact sequence (we use the notation $F_0^p := \tilde{X}^{\geq p}$ and $F_1^p := \mathfrak{R}_{\geq p}$ for the filtration)

$$\begin{array}{ccccccccc}
0 & \to & F_1^{p+1}/F_1^{p+2} & \longrightarrow & F_1^p/F_1^{p+2} & \longrightarrow & F_1^p/F_1^{p+1} & \to & 0 \\
 & & \downarrow{\scriptstyle \partial_A} & & \downarrow{\scriptstyle \partial_K^{Ind}} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \to & F_0^{p+1}/F_0^{p+2} & \longrightarrow & F_0^p/F_0^{p+2} & \longrightarrow & F_0^p/F_0^{p+1} & \to & 0
\end{array}$$

where $\partial_K^{Ind}$ denotes the induced map.

We extend the above diagram by adding a row of kernels at the top, and a row of cokernels at the bottom:

$$\begin{array}{ccccccccc}
 & & [ker\ \partial_A]_{p+1} & \longrightarrow & ker\ \partial_K^{Ind} & \xrightarrow{\rho} & [ker\ \partial_A]_p & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & F_1^{p+1}/F_1^{p+2} & \longrightarrow & F_1^p/F_1^{p+2} & \longrightarrow & F_1^p/F_1^{p+1} & \to & 0 \\
 & & \downarrow{\scriptstyle \partial_A} & & \downarrow{\scriptstyle \partial_K^{Ind}} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \to & F_0^{p+1}/F_0^{p+2} & \longrightarrow & F_0^p/F_0^{p+2} & \longrightarrow & F_0^p/F_0^{p+1} & \to & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & coker\ \partial_A & \xrightarrow{\nu} & coker\ \partial_K^{Ind} & \longrightarrow & A^p & &
\end{array}$$

Then, as before, the following sequence is exact by the Snake Lemma:

$$ker\ \partial_K^{Ind} \xrightarrow{\rho} [ker\ \partial_A]_p \xrightarrow{d_1} coker\ \partial_A \xrightarrow{\nu} coker\ \partial_K^{Ind}$$

and this $d_1$ is the differential for the page $E_1$. It is clear from the exact sequence that $\rho$ is injective iff $\nu$ is injective iff $d_1 = 0$. Unfortunately, though, this is not enough to prove the PVH Criterion, since in particular $ker\ \partial_K^{Ind}$ are not the syzygies of $K$, and $coker\ \partial_K^{Ind}$ is not $\frac{I_F^p}{\partial_K \mathfrak{R}_{\geq p}}$ (i.e. effectively because we have not considered all higher degree differentials of the spectral sequence).

However, we can bring all of the higher degree differentials in simultaneously by considering instead the coboundary operator arising from the exact sequence:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_1^{p+1} & \longrightarrow & F_1^p & \longrightarrow & F_1^p/F_1^{p+1} & \to & 0 \\
 & & \downarrow{\scriptstyle \partial_K^{res}} & & \downarrow{\scriptstyle \partial_K} & & \downarrow{\scriptstyle \partial_A} & & \\
0 & \longrightarrow & F_0^{p+1} & \longrightarrow & F_0^p & \longrightarrow & F_0^p/F_0^{p+1} & \to & 0
\end{array}$$

[5] See e.g. [Lang], Proposition XX 9.2, page 817.

and the result is just the diagram considered in our proof of Theorem 1. Moreover, the fact that $\pi^{Syz}$ is a surjection is equivalent to the coboundary operator from the above sequence being 0, i.e. there are no corrections to the $E_1$ page in computing $E_\infty$.

## 2.3 Some Reminders About Quadratic Duality

### 2.3.1 Basics

In this subsection we briefly review the theory of quadratic algebras to the extent needed to prove the final claim in Theorem 1, and to cover material that will be needed later (but skipping proofs). The reader who is not familiar with this theory can find a quick overview in [Fröberg2] or [Hille], or more extensive treatment in [Pol] and [Kraehmer]; the original source is [Priddy].[6]

We start with the quadratic algebra $A$ which is given by $A = TV/\langle R\rangle$ (in the notation of Subsection 1.1.6). The quadratic dual algebra $A^!$ is defined as $A^! := TV^*/\langle R^\perp\rangle$, where $V^*$ is the linear dual vector space and $R^\perp \subseteq V^* \otimes V^*$ is the annihilator of $R$.

One indication of the usefulness of the concept of quadratic duality is that the degree 2 part of the dual algebra catalogues the relations of the original algebra (this is true for all quadratic algebras). More generally, the Koszul complex provides a readily computable 'candidate' resolution for $A$, which is an actual resolution precisely when $A$ is Koszul. In particular the degree 3 part of the dual provides at least a candidate catalogue of the relations among the relations of the original algebra (i.e. syzygies) - and more generally, the degree $m$ part of the dual provides a candidate catalogue of the relations among relations among ... ($(m-1)$ times) of the original algebra (which we will call the level $m$ syzygies). Moreover, there are specific maps from the degree $m$ part of the dual into the space of level $m$ syzygies. The statement that a quadratic algebra is Koszul is equivalent to the statement that the dual algebra not only provides a candidate catalogue of the syzygies of all levels, but an actual, complete catalogue of those syzygies. For purposes of this paper, it is only the level 3 syzygies that are important.

More specifically, if we define $\Delta^!_{1,1} : A^{!2*} \to V \otimes V$ as the dual to multiplication $V^* \otimes V^* \to A^{!2}$, then in fact $\Delta^!_{1,1}$ is an isomorphism:

$$\Delta^!_{1,1} : \; A^{!2*} \xrightarrow{\sim} \; R \tag{10}$$

Thus $A^{!2}$ catalogues the degree 2 relations of $A$ and the map $\Delta^!_{1,1}$ sends a basis of $A^{!2}$ to a basis of $R$ (see (19) and (20) below, in the case of $\mathfrak{pvb}^!_n$).

[6] As noted in footnote 2, we rely on results about Koszul algebras which have only been developed for graded algebras whose graded components are finitely generated over the ground ring. Hence, wherever we rely on Koszulness of $A$, we assume the algebra $K$ to be finitely generated. This is sufficient to ensure that $A^m$ is finitely generated over $\mathbb{Q}$.

In the same vein, $A^{!3}$ catalogues all relations between relations of $A$, in degree three[7] - in other words, $A^{!3} \cong (R \otimes V \cap V \otimes R)$ (see [Pol], proof of Theorem 4.4.1). More specifically, if $\Delta^!_{2,1}$ is dual to the multiplication: $A^{!2} \otimes V^* \twoheadrightarrow A^{!3}$, then the map

$$(\Delta^!_{1,1} \otimes 1) \circ \Delta^!_{2,1} : \quad A^{!3*} \hookrightarrow \ R \otimes V \subseteq X^3_1 \tag{11}$$

is actually an isomorphism $A^{!3*} \xrightarrow{\sim} (R \otimes V \cap V \otimes R)$ which maps a basis for $A^{!3}$ to a basis for the degree 3 syzygies (viewed as a subspace of $X^3_1$).

Similarly, if $\Delta^!_{1,2}$ is dual to the multiplication: $V^* \otimes A^{!2} \to A^{!3}$, the map:

$$(1 \otimes \Delta^!_{1,1}) \circ \Delta^!_{1,2} : \quad A^{!3*} \hookrightarrow \ R \otimes V \subseteq X^3_2 \tag{12}$$

is an isomorphism $A^{!3*} \xrightarrow{\sim} (R \otimes V \cap V \otimes R)$, and maps a basis for $A^{!3}$ to a basis for the degree 3 syzygies (viewed as a subspace of $X^3_2$).

*A priori*, $A^{!3}$ need not generate the (level 3) syzygies of $A$ in degrees higher than 3. However, if $A$ is Koszul then indeed $A^{!3}$ *does* generate the (level 3) syzygies of $A$ in all degrees, as we will explain further in the next subsection.

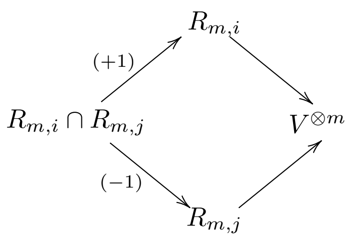
### 2.3.2 The Role of Koszulness

We will make use of the following theorem, which follows from [Pol], Theorem 2.4.1 (p.29) and Proposition 1.7.2 (p.16), to which the reader is referred for proofs.

**Theorem 2.** *Koszulness*[8] *of the algebra A implies exactness of the sequence:*

$$\bigoplus_{i<j} (R_{m,i} \cap R_{m,j}) \xrightarrow{\partial_{Syz}} \bigoplus_{i} R_{m,i} \xrightarrow{\kappa} V^{\otimes m} \tag{13}$$

In (13), the direct sums are external, and the maps are induced from the following diagram:




[7] If we assume that $\partial_A : \mathbb{Q}\mathcal{Y} \to R$ is injective (i.e. the PVH Criterion is satisfied in degree 2) then level 3 syzygies must have at least degree 3 in the generators of $A$. Given a level 3, degree 3 syzygy, we can also get level 3 syzygies of higher degree by pre- or post-multiplying all terms in the syzygy by monomials in the generators, although level 3 syzygies of higher degree need not all arise in this way (except when the algebra is Koszul).

[8] As per footnote 3, $A$ need only be 2-Koszul, i.e. its Koszul complex need only be exact up to homological degree 2 inclusive.

where the left diagonals are multiplication by the indicated factors, and the right diagonals are the inclusions.

Note that we can decompose $\bigoplus_{i<j}(R_{m,i} \cap R_{m,j})$ as follows:

$$\bigoplus_{i<j}(R_{m,i} \bigcap R_{m,j}) = \bigoplus_{i}(R_{m,i} \cap R_{m,i+1}) \oplus \bigoplus_{i+1<j}(R_{m,i} \cap R_{m,j})$$

The syzygies $\bigoplus_{i+1<j}(R_{m,i} \cap R_{m,j})$ are 'trivial' in the sense that they arise from the obvious fact that non-overlapping relations commute. This fact remains true at the global level, so that these 'trivial' syzygies also trivially satisfy the PVH Criterion.

The more interesting syzygies are the $(R_{m,i} \cap R_{m,i+1})$. From the review given in the previous subsection, we have $(R_{m,i} \cap R_{m,i+1}) \cong V^{\otimes i} \otimes (X_1^3 \cap X_2^3) \otimes V^{\otimes m-i-2} \cong V^{\otimes i} \otimes A^{!3} \otimes V^{\otimes m-i-2}$. This makes clear that the PVH Criterion need only be checked in degree 3 in the Koszul case.

# 3 The Quadraticity of $PvB_n$

## 3.1 Overview

We now turn to $PvB_n$. Our goal being to establish that $PvB_n$ is quadratic using the PVH Criterion, we will follow the following steps:

- Check that the preliminary requirements (as per Subsection 1.1.2) for applying the PVH Criterion are met.
- Find the infinitesimal syzygies. We will use the fact that $\mathfrak{pvb}_n$ is Koszul, and that accordingly the infinitesimal syzygies are essentially given by $\mathfrak{pvb}_n^{!3}$. (The Koszulness of $\mathfrak{pvb}_n^!$ was first established in [BarEnEtRa], and we give an alternative proof in subsection 3.7). After finding a basis for $\mathfrak{pvb}_n^!$, and in particular for $\mathfrak{pvb}_n^{!3}$, we will see that finding the infinitesimal syzygies becomes a fairly straightforward calculation.
- Find the global syzygies corresponding to the Zamolodchikov tetrahedron, and compute the induced infinitesimal syzygies.
- Check that global syzygies induce all of the infinitesimal syzygies, confirming that the PVH Criterion is met.

## 3.2 Terminology and Preliminary Requirements for PVH Criterion

We denote by $\mathbb{Q}PvB_n$ and $\mathbb{Q}F$ the rational group ring of $PvB_n$ and the rational free group ring on the same generators, respectively. Their respective augmentation ideals are denoted $I_K$ and $I_F$.

Consistent with the discussion in Subsection 2.1, we take $\mathbb{Q}PvB_n$ to be completed with respect to the filtration by powers of the augmentation ideal, so that we can eliminate the inverses of group generators, and the linear relations corresponding to the group laws, from our presentation for $\mathbb{Q}PvB_n$ as an algebra.

Given the presentation for $PvB_n$ in Section 1, the augmentation ideal $I_K$ is generated as a 2-sided $\mathbb{Q}PvB_n$-module by the set $\tilde{X} := \{\overline{R}_{ij} := (R_{ij} - 1) : 1 \leq i \neq j \leq n\}$. It is straightforward to check that the elements of $\tilde{X}$ (modulo $I_K^2$) are linearly independent (i.e. $\mathbb{Q}\tilde{X} \cap I_K^2 = 0$), and hence in fact form a basis of $V = I_K/I_K^2$. The $\overline{R}_{ij}$ mod $I_K^2$ correspond to the generators $\{r_{ij}\}$ for $\mathfrak{pvb}_n$ from the presentation (3).

From the relations (1) and (2) for $PvB_n$, the ideal $M \subseteq I_F$ of relations for $K$ is the 2-sided ideal $M$ in $F$ generated by

$$\begin{aligned} Y'_{ijk} &:= R_{ij}R_{ik}R_{jk}R_{ij}^{-1}R_{ik}^{-1}R_{jk}^{-1} - 1 \\ C_{ij}^{kl\prime} &:= R_{ij}R_{kl}R_{ij}^{-1}R_{kl}^{-1} - 1 \end{aligned}$$

for $1 \leq i, j, k, l \leq n$, and $i, j, k, l$ all distinct. Equivalently, $M$ is generated (as 2-sided $F$-ideal) by

$$Y_{ijk} := R_{ij}R_{ik}R_{jk} - R_{jk}R_{ik}R_{ij} \tag{14}$$

$$C_{ij}^{kl} := R_{ij}R_{kl} - R_{kl}R_{ij} \tag{15}$$

As per Lemma 1, the relations in $\mathfrak{pvb}_n$ are generated by $R \cong (M + I_F^3)/I_F^3$. Thus to obtain $R$, we make the substitution $R_{ij} \mapsto (\overline{R}_{ij} + 1)$ throughout the $Y_{ijk}$ and $C_{ij}^{kl}$, and drop all terms of degree 3 (or higher) in the $\overline{R}_{ij}$. We obtain the quadratic relators $\{y_{ijk}; c_{ij}^{kl}\}$ for $\mathfrak{pvb}_n$ (see (3) and (4)), up to replacing the $\{\overline{R}_{ij}\}$ by the $\{r_{ij}\}$.

Since $PvB_n$ is a finitely presented group, the requirements for applicability of the PVH Criterion, as we have developed it, essentially reduce to (see Subsection 1.1.2) checking that the ideal of relations $M \subseteq I_F$ actually satisfies $M \subseteq I_F^2$. In turn this amounts to checking that the relators obtained above for $R$ (i.e. (3)) are all quadratic in the $r_{ij}$, which is clearly true. We note that this is essentially due to the fact that the relators for $PvB_n$ all have degree 0 in each of the generators of $PvB_n$, so that after performing the substitution $R_{ij} \mapsto (\overline{R}_{ij}+1)$ and expanding in terms of the $\overline{R}_{ij}$, all constant terms and terms linear in the $\overline{R}_{ij}$ cancel out.

We can also easily check that $PvB_n$ satisfies the degree 2 PVH Criterion, i.e. that the generators (14) and (15) for $M$ satisfy the requirement that $\{Y_{ijk} + I_F^3, C_{ij}^{kl} + I_F^3\}$ are linearly independent in $(M + I_F^3)/I_F^3$. Equivalently we have to check that the $\{y_{ijk}; c_{ij}^{kl}\}$ are linearly independent. There are several ways to do this - one slightly fancy way to do it is to use the isomorphism

$$\Delta_{1,1}^! : \mathfrak{pvb}_n^{!2*} \xrightarrow{\sim} R$$

which we recalled in (10), and note that $\Delta_{1,1}^!$ takes a basis of $\mathfrak{pvb}_n^{!2}$ precisely to the relators $\{y_{ijk}; c_{ij}^{kl}\}$ of $\mathfrak{pvb}_n$ (we compute this in (19) and (20) below).

## 3.3 Finding the Infinitesimal Syzygies

As a preliminary matter we recall the definition of $\mathfrak{pvb}^!_n$ and exhibit its relations. As noted in Subsection 1.1.6, $\mathfrak{pvb}_n$ is defined as $\mathfrak{pvb}_n = TV/\langle R\rangle$ (where $V = I_K/I_K^2$ and $R$ were obtained in Subsection 3.2). The quadratic dual algebra $\mathfrak{pvb}^!_n$ is defined as $\mathfrak{pvb}^!_n := TV^*/\langle R^\perp\rangle$, where $V^*$ is the linear dual vector space and $R^\perp \subseteq V^* \otimes V^*$ is the annihilator of $R$.

From these definitions, one readily finds that $\mathfrak{pvb}^!_n$ is the exterior algebra generated by the set $\{r^*_{ij} : 1 \le i \ne j \le n\}$, subject to the relations:

$$r^*_{ij} \wedge r^*_{ik} = r^*_{ij} \wedge r^*_{jk} - r^*_{ik} \wedge r^*_{kj} \tag{16}$$

$$r^*_{ik} \wedge r^*_{jk} = r^*_{ij} \wedge r^*_{jk} - r^*_{ji} \wedge r^*_{ik} \tag{17}$$

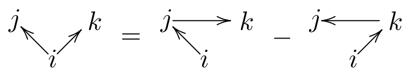
$$r^*_{ij} \wedge r^*_{ji} = 0 \tag{18}$$

where the indices $i, j, k$ are all distinct.

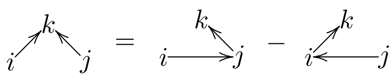
### 3.3.1 A Basis for $\mathfrak{pvb}^!_n$

In this section we will identify a basis for the dual algebra $\mathfrak{pvb}^!_n$. We state the result for all degrees, although we only actually need $\mathfrak{pvb}^{!3}_n$.

We note that monomials in $\mathfrak{pvb}^!_n$ may be interpreted as directed graphs, with vertices given by the integers $[n] := \{1, \ldots, n\}$, and edges consisting of all ordered pairs $(i, j)$ such that $r_{ij}$ is in the monomial. We thus get a graphical depiction of the above relations:




(Pruning V)




(Pruning A)


(No Loop)

We note that there is a sign indeterminacy in the graphs, in that for instance the LHS of (Pruning V) can equally refer to $\pm r_{ij} \wedge r_{ik}$. We will only use the graphs when the signs are immaterial.

**Theorem 3.** *The algebra $\mathfrak{pvb}^!_n$ has a basis consisting exactly of the monomials corresponding to 'chain gangs', i.e. unordered partitions of $[n]$ into ordered subsets.*

**Corollary 1.** *The degree $k$ component of $\mathfrak{pvb}^!_n$ has dimension $L(n, n-k)$, where the 'Lah number' $L(n, n-k)$ denotes the number of unordered partitions of $[n]$ into $(n-k)$ ordered subsets.*

*Proof.* Clear from the theorem, since it is easy to see that a chain gang on the index set $[n]$ with $(n-k)$ chains must have exactly $k$ arrows (and correspond to a basis monomial of degree $k$). □

We note that it was already proved in [BarEnEtRa] that the dimensions of the graded components of $\mathfrak{pvb}_n^!$ are given by the Lah numbers (although a basis for $\mathfrak{pvb}_n^!$ was not provided).

We postpone the proof until Subsection 3.5. However, the idea of the proof is straightforward, i.e. show that a basis is given by all monomials whose graphical representation has no **A-joins** or **V-joins** (by which we mean the diagrams in the LHS of the relations (Pruning A) and (Pruning V), respectvely) and no loops:

- One first shows that if a tree has an A-join or a V-join, we can replace it by a sum of trees in which the particular join is replaced by an oriented segment of length 2, using either (Pruning A) or (Pruning V). Eventually we are left with a sum of oriented chains.
- One must then show that these oriented chains are linearly independent.
- Next one shows that all monomials whose graph contains a loop (oriented or not) are 0: it turns out that loops of length greater than 2 can be reduced progressively to loops of length 2, and then the resulting graph is 0 either by (No Loops) or by anti-commutativity.

**Remark 5.** *We will see that directed chains of length 3 are in a 1-1 correspondence with certain (level 3) syzygies of the global algebra $\mathfrak{A}$ - specifically one Zamolodchikov tetrahedron for each ordering of a particular choice of 4 of the n strands in $PvB_n$. An arrow from index 'i' to index 'j' means strand 'i' remains above strand 'j' throughout the syzygy. Although not relevant for our purposes, this correspondence between oriented chains of length m and level m syzygies holds for syzygies of all levels. These higher level global syzygies correspond to generalizations of the Zamolodchikov tetrahedron, and most likely correspond in some sense to generators of the cohomology of $PvB_n$.*

**Remark 6.** *If in the basis given above one includes only generators $r_{ij}$ with $i<j$, we obtain a basis for the algebra $\mathfrak{pfb}_n^!$. This basis is different from the basis given in [BarEnEtRa]. The basis given here is more useful for purposes of applying the PVH criterion, because of the fact that directed chains of length 3 correspond to syzygies arising from the Zamolodchikov tetrahedron.*

**Remark 7.** *If, as in the previous remark, we again consider the implied basis for $\mathfrak{pfb}_n^!$, we see that the (No Loop) relation and the exclusion of monomials whose graph contains a loop are irrelevant. We are left with a rule that says that a basis of $\mathfrak{pfb}_n^!$ is given by all monomials whose graph does not contain an A-join or a V-join. The exclusion of A-joins and V-joins amounts to specifying a quadratic Gröbner basis for the ideal of relations in $\mathfrak{pfb}_n^!$. By a theorem of [Yuz], this gives a proof that the algebra $\mathfrak{pfb}_n^!$ (and its dual $\mathfrak{pfb}_n$) is Koszul. Unfortunately the given basis for $\mathfrak{pvb}_n^!$, as opposed to $\mathfrak{pfb}_n^!$, does not prove Koszulness, since* *the no-loop exclusion corresponds to Gröbner basis elements of arbitrarily high degree (i.e. of degree equal to the length of the loop). In subsection 3.7 we give an alternative basis for* $\mathfrak{pvb}_n^!$*, from which the Koszulness of* $\mathfrak{pvb}_n^!$ *can be deduced.*

### 3.3.2 The Infinitesimal Syzygies

One can readily compute that the isomorphism $\Delta_{1,1}^!$ acts on basis elements of $\mathfrak{pvb}_n^{!2*}$ as follows:[9]

$$\Delta_{1,1}^! : r_{ij} \wedge r_{jk} \mapsto [r_{ij}, r_{ik}] + [r_{ij}, r_{jk}] + [r_{ik}, r_{jk}] \tag{19}$$

$$r_{ij} \wedge r_{kl} \mapsto [r_{ij}, r_{kl}] \tag{20}$$

Indeed one can in fact view (Pruning A) and (Pruning V) as giving the only elements of $V^*$ that do not multiply freely in $\mathfrak{pvb}_n^{!2}$, and then (since $\Delta_{1,1}^!$ is dual to the product in $\mathfrak{pvb}_n^{!2}$) the above result is immediate.

As noted following (11) and (12), the maps $(1\otimes\Delta_{1,1}^!)\circ\Delta_{1,2}^!$ and $(\Delta_{1,1}^!\otimes 1)\circ\Delta_{2,1}^!$ are isomorphisms and give the inclusion of $\mathfrak{pvb}_n^{!3}$ into $X_1^3$ and $X_2^3$ respectively. It follows that the image of the map $\kappa\circ\partial_{Syz}$ of $X_1^3\cap X_2^3$ into $V^{\otimes 3}$ (see (13)) is given by

$$\kappa\circ\partial_{Syz}(X_1^3\cap X_2^3) = \big[(\Delta_{1,1}^!\otimes 1)\circ\Delta_{2,1}^! - (1\otimes\Delta_{1,1}^!)\circ\Delta_{1,2}^!\big](\mathfrak{pvb}_n^{!3}) \tag{21}$$

It is easy to see that there are three types of basis element in $\mathfrak{pvb}_n^{!3}$, corresponding to three types of chain gang with three edges:

- $r_{ij}\wedge r_{jk}\wedge r_{kl}$ with $i,j,k,l$ all distinct;
- $r_{ij}\wedge r_{jk}\wedge r_{st}$ with $i,j,k,s,t$ all distinct;
- $r_{ij}\wedge r_{kl}\wedge r_{st}$ with $i,j,k,l,s,t$ all distinct.

We first deal with the first type of basis element. We will show that in this case the first term of (21) is given by:[10]

$$\begin{aligned}
\Delta_{2,1}^!(r_{ij}\wedge r_{jk}\wedge r_{kl}) = &-(\Delta_{1,1}^!\otimes 1)(r_{ij}\wedge r_{jk})\otimes(-r_{il}-r_{jl}-r_{kl})\\
&+(\Delta_{1,1}^!\otimes 1)(r_{ij}\wedge r_{jl})\otimes(-r_{ik}-r_{jk}+r_{kl})\\
&-(\Delta_{1,1}^!\otimes 1)(r_{ik}\wedge r_{kl})\otimes(-r_{ij}+r_{jk}+r_{jl})\\
&+(\Delta_{1,1}^!\otimes 1)(r_{jk}\wedge r_{kl})\otimes(r_{ij}+r_{ik}+r_{il})\\
&-(\Delta_{1,1}^!\otimes 1)(r_{ij}\wedge r_{kl})\otimes(r_{ik}+r_{il}+r_{jk}+r_{jl})\\
&+(\Delta_{1,1}^!\otimes 1)(r_{ik}\wedge r_{jl})\otimes(r_{ij}+r_{il}-r_{jk}+r_{kl})\\
&+(\Delta_{1,1}^!\otimes 1)(r_{il}\wedge r_{jk})\otimes(-r_{ij}-r_{ik}+r_{jl}+r_{kl})
\end{aligned}$$

[9] Instead of writing $r_{ij}^{**}$ for elements of $\mathfrak{pvb}_n^{!2*}$, we write $r_{ij}$.

[10] Again, we write $r_{ij}$ instead of $r_{ij}^{**}$ for elements of $\mathfrak{pvb}_n^{!3*}$.

We defer a more detailed justification of the above calculation to Subsection 3.6. Now using (19) and (20), we get:

$$\begin{aligned}(\Delta^!_{1,1}\otimes 1)\circ\Delta^!_{2,1}(r_{ij}\wedge r_{jk}\wedge r_{kl}) &= -y_{ijk}\otimes(-r_{il}-r_{jl}-r_{kl})+y_{ijl}\otimes(-r_{ik}-r_{jk}+r_{kl})\\ &-y_{ikl}\otimes(-r_{ij}+r_{jk}+r_{jl})+y_{jkl}\otimes(r_{ij}+r_{ik}+r_{il})\\ &-c^{kl}_{ij}\otimes(r_{ik}+r_{il}+r_{jk}+r_{jl})+c^{jl}_{ik}\otimes(r_{ij}+r_{il}-r_{jk}+r_{kl})\\ &-c^{jk}_{il}\otimes(r_{ij}+r_{ik}-r_{jl}-r_{kl}) \qquad (22)\end{aligned}$$

In (22) the tensor products are the tensor products in the tensor algebra $TV$, so we drop them. Furthermore, $(1\otimes\Delta^!_{1,1})\circ\Delta^!_{1,2}(r_{ij}\wedge r_{jk}\wedge r_{kl})$ is the same, but with the tensor components flipped. Putting the two together gives:

$$\begin{aligned}(\Delta^!_{1,1}\otimes 1)\circ\Delta^!_{2,1}(r_{ij}\wedge r_{jk}\wedge r_{kl}) &= -[y_{ijk},(-r_{il}-r_{jl}-r_{kl})]+[y_{ijl},(-r_{ik}-r_{jk}+r_{kl})]\\ &-[y_{ikl},(-r_{ij}+r_{jk}+r_{jl})]+[y_{jkl},(r_{ij}+r_{ik}+r_{il})]\\ &-[c^{kl}_{ij},(r_{ik}+r_{il}+r_{jk}+r_{jl})]+[c^{jl}_{ik},(r_{ij}+r_{il}-r_{jk}+r_{kl})]\\ &-[c^{jk}_{il},(r_{ij}+r_{ik}-r_{jl}-r_{kl})] \qquad (23)\end{aligned}$$

We will see that these syzygies are induced (via the map $\pi^{Syz}$) from global syzygies in the next subsection.

This leaves the two remaining types of degree 3 basis element in $\mathfrak{pvb}_n^{!3}$. It is fairly straightforward to compute that they correspond, respectively, to the relations:

$$y_{ijk}r_{st}=r_{st}y_{ijk}$$

and

$$c^{kl}_{ij}r_{st}=r_{st}c^{kl}_{ij}$$

which are clearly satisfied also at the global level.

## 3.4 Global Syzygies and the PVH Criterion

### 3.4.1 The Global Syzygies

We now display (a sum of) elements of $\mathfrak{R}_3$ which specify a syzygy of $K=\mathbb{Q}PvB_n$ and which project via $\pi^{Syz}$ (see notation in Theorem 1) to the syzygy (23) in $\mathfrak{pvb}_n$. The syzygy corresponds to the standard syzygy in the braid group (i.e. the Zamolodchikov tetrahedron pictured in Subsection 1.2), which can be written:

$$\begin{aligned}&Y_{jkl}R_{il}R_{ik}R_{ij}+R_{jk}R_{jl}Y_{ikl}R_{ij}+R_{jk}R_{jl}R_{ik}R_{il}C^{kl}_{ij}\\ &+R_{jk}C^{jl}_{ik}R_{il}R_{ij}R_{kl}+R_{jk}R_{ik}Y_{ijl}R_{kl}+Y_{ijk}R_{il}R_{jl}R_{kl}+R_{ij}R_{ik}C^{jk}_{il}R_{jl}R_{kl}\\ &-R_{ij}R_{ik}R_{il}Y_{jkl}-R_{ij}Y_{ikl}R_{jl}R_{jk}-C^{kl}_{ij}R_{il}R_{ik}R_{jl}R_{jk}\\ &-R_{kl}R_{ij}R_{il}C^{jl}_{ik}R_{jk}-R_{kl}Y_{ijl}R_{ik}R_{jk}-R_{kl}R_{jl}R_{il}Y_{ijk}-R_{kl}R_{jl}C^{jk}_{il}R_{ik}R_{ij}\end{aligned}$$

where again

$$Y_{ijk} = R_{ij}R_{ik}R_{jk} - R_{jk}R_{ik}R_{ij} \tag{24}$$
$$C_{ij}^{kl} = R_{ij}R_{kl} - R_{kl}R_{ij}$$

This calculation was illustrated in Subsection 1.2.

The calculation may be explained as follows. The illustration shows 14 braids $\{B_i\}_{i=1,\dots,14}$ around its perimeter. These are linked by arrows labeled by various multiples of the moves $Y_{ijk}$ or $C_{ij}^{kl}$. If we attach the labels $B_1, B_2, \dots$ starting at the bottom braid and proceeding clockwise around the perimeter, the arrows correspond to differences $(B_2 - B_1), \dots, (B_8 - B_7)$ up the left side of the diagram, and to differences $(B_{14} - B_1), \dots, (B_8 - B_9)$ around the right side. If we label the differences in accordance with (24), we get the labeling in the picture. But clearly the telescopic sums on the left and right both give $B_8 - B_1$, so we get a syzygy which we wrote down above.

This syzygy induces an infinitesimal syzygy which we obtain by substituting $R_{ij} \mapsto (\overline{R}_{ij} + 1)$ and dropping all but the lowest degree terms in the $\overline{R}_{ij}$ (this corresponds to applying the map $\pi^{Syz}$). After reorganizing, we get:

$$\begin{aligned}[]
[Y_{jkl}, \overline{R}_{ij} + \overline{R}_{ik} + \overline{R}_{il}] - [Y_{ikl}, -\overline{R}_{ij} + \overline{R}_{jk} + \overline{R}_{jl}] + [Y_{ijl}, -\overline{R}_{ik} - \overline{R}_{jk} + \overline{R}_{kl}] \\
- [Y_{ijk}, -\overline{R}_{il} - \overline{R}_{jl} - \overline{R}_{kl}] \\
- [C_{ij}^{kl}, \overline{R}_{ik} + \overline{R}_{il} + \overline{R}_{jk} + \overline{R}_{jl}] + [C_{ik}^{jl}, \overline{R}_{ij} + \overline{R}_{il} - \overline{R}_{jk} + \overline{R}_{kl}] \\
- [C_{il}^{jk}, \overline{R}_{ij} + \overline{R}_{ik} - \overline{R}_{jl} - \overline{R}_{kl}]
\end{aligned}$$

where the $C_{ij}^{kl}$ and $Y_{ijk}$ are the same as the $c_{ij}^{kl}$ and $y_{ijk}$ in (3), except the $r_{ij}$ are replaced by the $\overline{R}_{ij}$. By inspection, and after substituting $\overline{R}_{ij} \mapsto r_{ij}$ and $\{C_{ij}^{kl} \mapsto c_{ij}^{kl},\ Y_{ijk} \mapsto y_{ijk}\}$, we see that this coincides with the infinitesimal syzygy (23). Hence we have confirmed that all of the infinitesimal syzygies are covered by global syzygies.

## 3.5 Proof of the Basis for $\mathfrak{pvb}_n^!$

We will follow the outline of the proof provided in Subsection 3.3.1.

We will say that a pair of vertices in a forest graph is **unordered** if there is not an oriented sequence of edges from one of them to the other. We define the **defect** of a tree as the number of unordered pairs of vertices in the graph, and the defect of a forest as the sum of the defects of its components.

Then chain gangs (unordered partitions of $[n]$ into ordered subsets) are exactly the forests with 0 defect. Moreover, in the pruning moves the A- and V-joins have defect 1, while the remaining terms have defect 0.

We will refer to a relation formed by adding to each of the terms in either (Pruning A) or (Pruning V) exactly the same additional edges, without ever forming a loop, as a **multiple** of the original relation. Note that the graphs representing the multiple need not be connected. The defect function has the following 'multiplicativity' property on forests:

**Lemma 4.** *In each multiple of either (Pruning A) or (Pruning V), the term which is built from the term in the original relation containing a join has defect strictly larger than the other terms.*

The proof is deferred to the end of this subsection.

*Proof of Theorem 3.* We follow the plan of proof given following the statement of Theorem 3.

The multiplicativity property of the defect makes clear that all forests can be expressed in terms of (sums of) chain gangs: If a forest contains an A- or V-join, then using either (Pruning A) or (Pruning V) we can replace it with a sum of forests with strictly lower defect. Iterating, we get a forest with 0 defect, i.e. a chain gang.

Now we show that chain gangs are linearly independent modulo the relations in $\mathfrak{pvb}^!_n$. The proof is a variation on the standard diamond lemma proof, which we briefly recall. By a **reduction**, we mean specifically replacing the LHS of either pruning relation, or a multiple thereof, by the RHS.

We will show that reducing a defect to 0 will produce the same chain gangs regardless of the sequence of reductions chosen, by induction on the size of the defect. This is clearly true when we start with a forest with defect 1, since there is only one way to reduce such a forest.

Suppose the claim is true for all forests of defect $\leq m$. Let us consider a forest of defect $m+1$, and suppose there are two possible reductions, called (a) and (b). Then applying either (a) or (b) gives a (sum of) new forests, which we call A and B respectively, each of defect $\leq m$.

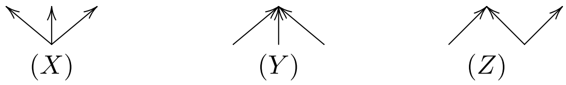
Suppose (a) and (b) (or the pruning relations of which they are multiples) involve changes to pairs of edges that do not overlap. Then it is still possible to apply reduction (a) to B, and reduction (b) to A. Doing so, we obtain the same forest C of defect $\leq m-1$, since the result of applying non-overlapping reductions clearly does not depend on the order they are applied.

Alternatively, suppose (a) and (b) (or the pruning relations of which they are multiples) involve changes to pairs of edges that do overlap. We will see that we can find further reductions (a'), (a") and (b'), (b") such that applying the sequence (a)-(a')-(a") or (b)-(b')-(b") leads to the same (sum of) forests C, of defect $\leq m-2$.[11]

Either way, we know by induction that all reduction sequences from A give the same results, and similarly for B, and since they have a common reduction sequence going through C, we see that A and B both give the same (sum of) forests of defect 0. Hence all reductions of the original forest must give the same (sum of) chain gangs.

We now deal with the case where reductions (a) and (b) involve pairs of edges that overlap, and exhibit the reductions (a'), (a") and (b'), (b"). By inspection of the A- and V-joins, the following three types of overlap can arise (up to sign):

[11] In fact the reductions (a') and (b') may really involve two reductions, applicable to different terms.



In each case we have only shown the edges involved in the reductions.

Case (Z) is dealt with as follows (a star over a wedge $\overset{*}{\wedge}$ indicates the join which is being reduced - hence to make the following more legible we have dropped the $^*$ from elements $r^*_{ij} \in \mathfrak{pvb}^!_n$):

$$\begin{aligned}
r_{ij} \overset{*}{\wedge} r_{kj} \wedge r_{kl} &= r_{ik} \wedge r_{kj} \overset{*}{\wedge} r_{kl} + r_{ij} \wedge r_{ki} \overset{*}{\wedge} r_{kl} \\
&= r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{ik} \wedge r_{kl} \wedge r_{lj} - r_{ij} \overset{*}{\wedge} r_{il} \wedge r_{ki} + r_{kl} \wedge r_{li} \wedge r_{ij} \\
&= r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{ik} \wedge r_{kl} \wedge r_{lj} - r_{ki} \wedge r_{ij} \wedge r_{jl} + r_{ki} \wedge r_{il} \wedge r_{lj} - r_{kl} \wedge r_{li} \wedge r_{ij}
\end{aligned}$$

while on the other hand

$$\begin{aligned}
r_{ij} \wedge r_{kj} \overset{*}{\wedge} r_{kl} &= r_{ik} \wedge r_{kj} \overset{*}{\wedge} r_{kl} + r_{ki} \overset{*}{\wedge} r_{kl} \wedge r_{ij} \\
&= r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{ik} \wedge r_{kl} \wedge r_{lj} + r_{ki} \wedge r_{il} \overset{*}{\wedge} r_{ij} - r_{kl} \wedge r_{li} \wedge r_{ij} \\
&= r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{ik} \wedge r_{kl} \wedge r_{lj} + r_{ki} \wedge r_{il} \wedge r_{lj} - r_{ki} \wedge r_{ij} \wedge r_{jl} - r_{kl} \wedge r_{li} \wedge r_{ij}
\end{aligned}$$

Since the results of the two calculations are the same, we see that regardless of which join we reduce first, there is a further sequence of reductions that leads to the same (signed sum of) trees, which is what we needed.[12]

Cases (X) and (Y) are dealt with similarly - we simply note that

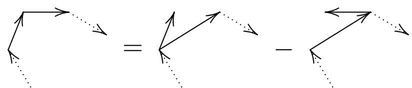
$$\begin{aligned}
r_{ij} \overset{*}{\wedge} r_{ik} \wedge r_{il} = &-r_{ij} \wedge r_{jl} \wedge r_{lk} + r_{ij} \wedge r_{jk} \wedge r_{kl} + r_{il} \wedge r_{lj} \wedge r_{jk} \\
&+ r_{ik} \wedge r_{kl} \wedge r_{lj} - r_{ik} \wedge r_{kj} \wedge r_{jl} - r_{il} \wedge r_{lk} \wedge r_{kj} = r_{ij} \wedge r_{ik} \overset{*}{\wedge} r_{il}
\end{aligned}$$

and

$$\begin{aligned}
r_{il} \overset{*}{\wedge} r_{jl} \wedge r_{kl} = &\; r_{ij} \wedge r_{jk} \wedge r_{kl} - r_{ik} \wedge r_{kj} \wedge r_{jl} + r_{ki} \wedge r_{ij} \wedge r_{jl} \\
&- r_{ji} \wedge r_{ik} \wedge r_{kl} + r_{jk} \wedge r_{ki} \wedge r_{il} - r_{kj} \wedge r_{ji} \wedge r_{il} = r_{il} \wedge r_{jl} \overset{*}{\wedge} r_{kl}
\end{aligned}$$

and leave the details of the calculations for the reader.

The next step in the proof is to show that all graphs with loops are 0. Let us start by considering oriented loops. Using (Pruning V), we can reduce oriented loops of length greater than 2 to (sums of) oriented loops of shorter length:

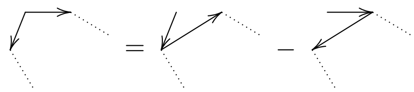


[12]As per the previous footnote, note that reductions (a') and (b'), indicated by the stars in the RHS of the first lines, actually involve two reductions, applicable to separate terms.

Once again we note that there is a sign indeterminacy in the above graphical representation, which does not affect the outcome as we do not rely on any cancelation of terms and the specific coefficients do not matter.

Once we are down to oriented loops of length 2, these are 0 by (No Loops).

Now we consider unoriented loops. For loops containing a V-join, we can use (Pruning V) to reduce loops of length greater than 2 to (sums of) loops of shorter length:



The case of unoriented loops containing an A-join, rather than a V-join, is similar. Although the result of any such reduction may or may not be unoriented, we can still continue reducing the length of the loops using either the oriented or unoriented procedure. Once we are down to loops of length 2, these are 0 either by (No Loops) or by anti-commutativity.

Thus, if we follow the above procedure, we can reduce all loops to 0. It is also clear from the above that even if we followed a different sequence of pruning moves we would never reduce loops to a sum of diagrams including trees, since a pruning move can never break a loop. □

We have completed the proof of Theorem 3, subject to proving multiplicativity of the defect function, i.e. Lemma 4. We do this now.

By a '**vertex in a relation**' we will mean a vertex which is an endpoint of at least one edge in the graphs corresponding to the terms of the relation. It is fairly clear this is a well-defined notion (and in particular that the number of vertices in a relation is constant over all terms in a relation).

By the '**join term**' in a multiple of a pruning relation, we mean the term that was built by adding edges to the term in the original pruning relation which contained an A- or V-join.

*Proof of Lemma 4.* We proceed by induction on the number of vertices in a relation. We claim that if (x) and (y) are vertices in the new relation, and there is a directed chain of edges from (x) to (y) in the join term, then there is also a directed chain of edges from (x) to (y) (in the same direction) in the other terms of the relation. Hence:

1. When we form a multiple of (Pruning A) or (Pruning V) by adding edges, in that multiple each vertex is no more ordered (with respect to other vertices) in the join-term than in the non-join terms.

2. However, in each relation, there is at least one pair of vertices which is unordered in the join-term, but is ordered in the other terms, namely the unordered paid in the original pruning relation.

So we can conclude that the join-term in the new relation must have strictly highest defect.

The above claim is easily verified in the original relations (Pruning A) and (Pruning V). We now assume the claim has been proved whenever there are up to $m$ edges in a relation; we take a relation with $m$ edges and add a further edge. There are three cases.

**Case I: Two New Vertices.** If the added edge forms a separate component in the new graphs, then clearly the defect will have increased by the same amount in all terms of the relation.

**Case II: One Old, One New Vertex.** So let us suppose that the added edge has one vertex (a) already in the relation, and one new vertex (b). It is clear that the orderliness of pairs of vertices not including (b) is unchanged.

Now suppose that (c) is any other vertex in the relation. If (b) and (c) are ordered in the new join term, say with a directed chain from (b) to (c), this chain must go through (a) since vertex (b) was not previously the endpoint of any edge. Thus there was also a directed chain from (a) to (c) in the join-term of the old relation, hence by induction there were directed chains from (a) to (c) in the non-join terms in the old relation. It follows that there is also a directed chain from (b) to (c) in the new non-join terms. The case of a directed chain from (c) to (b) in the new join-term is similar.

If (b) and (c) are unordered in the new join term, there is nothing to prove. Letting (c) range over all other vertices in the relation proves Case II.

**Case III: Two Old Vertices.** All that is left is to consider the case where the new edge links two existing vertices in the relation. Because we assume that the added edge does not create a loop, it follows that the edge must be linking two formerly disconnected components of the graphs underlying the relation. We assume the new edge links existing vertices (a) and (b). It is clear that the orderliness of pairs of vertices already within the same component in the old relation is unchanged. So we take (c) and (d) to be to be any two vertices in the component of (a) and (b) respectively. We can assume without loss of generality that either (a) $\neq$ (c) or (b) $\neq$ (d) (because if (a) = (c) and (b) = (d) then that pair is joined by the new edge and hence ordered in all terms of the relation).

The reasoning is similar to Case II. If (c) and (d) are ordered in the new join term, say with a directed chain from (c) to (d), this chain must go through (a) and (b) since we assume there are no loops. Thus the new edge must be oriented (a) to (b); moreover, there must also have been directed chains from (c) to (a) and from (b) to (d) in the join-term of the old relation. By induction, there were directed chains from (c) to (a) and from (b) to (d) in the non-join terms in the old relation. It follows that there is also a directed chain from (c) to (d) in the new non-join terms. The case of a directed chain from (d) to (c) in the new join-term is similar.

Finally, if (c) and (d) are unordered in the new join term, there is nothing to prove. Letting (c) and (d) range over all other vertices in the relation proves Case III. □

## 3.6 Justification of the Co-Product Formulas

We give here a summary of the action of the product $m^! : \mathfrak{pvb}_n^{!2} \otimes V^* \to \mathfrak{pvb}_n^{!3}$ in terms of the 'directed chains' basis for the respective spaces. The verifications are routine and we will leave them to the reader.

$$
\begin{aligned}
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{il} &\mapsto r^*_{il} \wedge r^*_{lj} \wedge r^*_{jk} - r^*_{ij} \wedge r^*_{jl} \wedge r^*_{lk} + r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{jl} &\mapsto -r^*_{ij} \wedge r^*_{jl} \wedge r^*_{lk} + r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{kl} &\mapsto r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{li} &\mapsto r^*_{li} \wedge r^*_{ij} \wedge r^*_{jk} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{lj} &\mapsto -r^*_{il} \wedge r^*_{lj} \wedge r^*_{jk} + r^*_{li} \wedge r^*_{ij} \wedge r^*_{jk} \\
r^*_{ij} \wedge r^*_{jk} \otimes r^*_{lk} &\mapsto r^*_{ij} \wedge r^*_{jl} \wedge r^*_{lk} - r^*_{il} \wedge r^*_{lj} \wedge r^*_{jk} + r^*_{li} \wedge r^*_{ij} \wedge r^*_{jk}
\end{aligned}
$$

and

$$
\begin{aligned}
r^*_{ij} \wedge r^*_{kl} \otimes r^*_{ik} &\mapsto -r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} + r^*_{ik} \wedge r^*_{kj} \wedge r^*_{jl} - r^*_{ik} \wedge r^*_{kl} \wedge r^*_{lj} \\
r^*_{ij} \wedge r^*_{kl} \otimes r^*_{ki} &\mapsto r^*_{kl} \wedge r^*_{li} \wedge r^*_{ij} - r^*_{ki} \wedge r^*_{il} \wedge r^*_{lj} + r^*_{ki} \wedge r^*_{ij} \wedge r^*_{jl} \\
r^*_{ij} \wedge r^*_{kl} \otimes r^*_{il} &\mapsto -r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} + r^*_{ik} \wedge r^*_{kj} \wedge r^*_{jl} - r^*_{ik} \wedge r^*_{kl} \wedge r^*_{lj} \\
&\quad - r^*_{ki} \wedge r^*_{ij} \wedge r^*_{jl} + r^*_{ki} \wedge r^*_{il} \wedge r^*_{lj} \\
r^*_{ij} \wedge r^*_{kl} \otimes r^*_{li} &\mapsto r^*_{kl} \wedge r^*_{li} \wedge r^*_{ij} \\
r^*_{ij} \wedge r^*_{kl} \otimes r^*_{jk} &\mapsto -r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} \\
r^*_{ij} \wedge r^*_{kl} \otimes r^*_{kj} &\mapsto r^*_{kl} \wedge r^*_{li} \wedge r^*_{ij} - r^*_{ki} \wedge r^*_{il} \wedge r^*_{lj} + r^*_{ki} \wedge r^*_{ij} \wedge r^*_{jl} \\
&\quad + r^*_{ik} \wedge r^*_{kl} \wedge r^*_{lj} - r^*_{ik} \wedge r^*_{kj} \wedge r^*_{jl} \\
r^*_{ij} \wedge r^*_{kl} \otimes r^*_{jl} &\mapsto -r^*_{ij} \wedge r^*_{jk} \wedge r^*_{kl} + r^*_{ik} \wedge r^*_{kj} \wedge r^*_{jl} - r^*_{ki} \wedge r^*_{ij} \wedge r^*_{jl} \\
r^*_{ij} \wedge r^*_{kl} \otimes r^*_{lj} &\mapsto r^*_{kl} \wedge r^*_{li} \wedge r^*_{ij} - r^*_{ki} \wedge r^*_{il} \wedge r^*_{lj} + r^*_{ik} \wedge r^*_{kl} \wedge r^*_{lj}
\end{aligned}
$$

Again the verification of the formulae for the dual map $\Delta^!_{2,1}$ is tedious but routine and is left to the (beleaguered) reader.

## 3.7 Proof of the Koszulness of $\mathfrak{pvb}_n^!$

As indicated in Remark 7, the basis given in Theorem 3 will not in itself lead to a proof of Koszulness because the explicit exclusion of monomials whose graphical representation contains a loop corresponds to Gröbner basis elements of arbitrarily high degree. In contrast, standard theorems on Gröbner bases only tell us that (under mild assumptions) algebras with quadratic Gröbner bases are Koszul.

So we will exhibit a different basis for $\mathfrak{pvb}_n^!$, consisting of all monomials not containing certain length two subwords, which corresponds to the specification of a quadratic Gröbner basis for $\mathfrak{pvb}^!_n$. We will see that, by a result of Yuzvinsky [Yuz] (see also [ShelYuz]), $\mathfrak{pvb}^!_n$(and hence also $\mathfrak{pvb}_n$) is Koszul.

To begin with, given any finite subset $I \subseteq \mathbb{N}$ (which we order numerically), we will define two kinds of graph with vertices indexed by $I$ - we will call these Down graphs and Up graphs. We will then show how to combine Down and Up graphs to get graphs (which we will call Up-Down graphs) which correspond to a new basis for $\mathfrak{pvb}^!_n$, of the desired form (i.e. corresponding to the specification of a quadratic Gröbner basis for $\mathfrak{pvb}^!_n$).

We will also see that the Down and Up graphs, respectively, catalogue bases for:

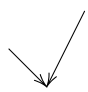
- the algebra $\mathfrak{pfb}^!_n$, which is quadratic dual to the quadratic approximation for the flat virtual braid group $PfB_n$ ( i.e. the quadratic dual to the universal enveloping algebra of the triangular Lie algebra $\mathfrak{tr_n}$ in [BarEnEtRa]); and
- the algebra $\mathfrak{pb}^!_n$, quadratic dual to the quadratic approximation for the pure braid group $PB_n$.

However, we do not have a coherent explanation for why bases for $\mathfrak{pfb}^!_n$ and $\mathfrak{pb}^!_n$ should fit together in this way to produce bases of $\mathfrak{pvb}^!_n$. See Remark 11 below.

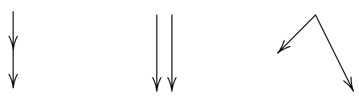
### 3.7.1 Down Graphs and $\mathfrak{pfb}^!_n$

A **Down tree** on the index set $I = \{i_1, \ldots, i_m\} \subseteq \mathbb{N}$ (with smallest index $i_1$) consists of a 'tuft' of directed edges $\{(i_2, i_1), \ldots, (i_m, i_1)\}$. (The graph is non-planar in that all orderings of the edges incident to a particular vertex are considered equivalent.) This corresponds to allowing all trees built with directed edges with *decreasing* indices (i.e. edges $(i, j)$ with $i > j$) by

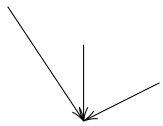
- allowing the following subgraphs:



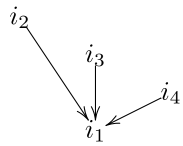
- excluding the following three subgraphs:



where in all cases the relative heights of the endpoints indicate the relative ordering of the indices (in particular, the middle subgraph has a doubled edge: $\{(i, j), (i, j)\}$). We declare by way of convention that a Down tree on on index set with one element is the empty graph. An example of a Down graph is the following:



i.e.




where $i_2 > i_3 > i_4 > i_1$.

Note that because of the last two types of excluded graph, we needn't have explicitly restricted ourselves to trees, as these exclusions prevent the formation of (ordered or unordered) loops in the graph (recall also that Down graphs are built only with directed edges with decreasing indices). Thus we have an exclusion rule, of degree 2 in the number of edges, which effectively eliminates loops. In particular the obstacle to proving Koszulness due to the presence of non-quadratic Gröbner basis elements no longer arises.

We now define a Down forest on an index set $I$ partitioned as $I = S_1 \sqcup \cdots \sqcup S_u$ to be the union of the Down trees on the subsets $S_i$.

**Remark 8.** *The monomials corresponding to Down forests induced by unordered partitions of* $[n] = \{1, 2, \ldots, n\}$ *(using the correspondence explained in subsection 3.3.1) form a basis for the algebra* $\mathfrak{pfb}^!_n$ *(as a skew-commutative algebra* [13]*). Indeed, it is easy to see that Down forests are in bijective correspondence with the 'reduced monomials with disjoint supports' which were proved in [BarEnEtRa], Proposition 4.2, to form a basis of* $\mathfrak{pfb}^!_n = U(\mathfrak{tr}_\mathfrak{n})^!$ *(with the minor difference that the edges in [BarEnEtRa] had increasing indices). Also, the above excluded subgraphs correspond to the excluded monomials implied by the Gröbner basis given in [BarEnEtRa], Corollary 4.3, for* $U(\mathfrak{tr}_\mathfrak{n})^!$ *(subject to always writing generators with increasing indices, using the relation* $r_{i,j} = -r_{j,i}$*). The fact that these Gröbner basis elements are quadratic allowed [BarEnEtRa] to conclude that* $\mathfrak{pfb}^!_n$ *is Koszul.*

### 3.7.2 Up Graphs and $\mathfrak{pb}^!_n$

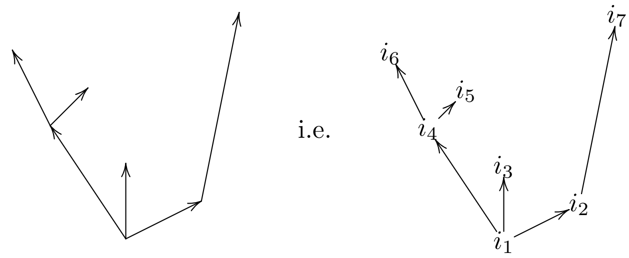
An **Up tree** on the index set $I = \{i_1, \ldots, i_m\} \subseteq \mathbb{N}$ (with $i_1 < \cdots < i_m$) consists of all trees built with directed edges with *increasing* indices by

- allowing the following two subgraphs:

- excluding the following two subgraphs:

[13] See [Mikha] for more on such bases.

where, again, in all cases the relative heights of the endpoints indicate relative ordering of the indices. Furthermore, the graphs are again non-planar in that all orderings of the edges incident to a particular vertex are considered equivalent; also, we again declare by way of convention that a Up tree on an index set with one element is the empty graph. An example of a Up tree is the following:




where $i_1 < \cdots < i_7$.

As with Down graphs we needn't have explicitly restricted ourselves to trees, since one effect of the excluded subgraphs is to prevent the formation of (ordered or unordered) loops in the graph. Again, the obstacle to proving Koszulness due to the presence of non-quadratic Gröbner basis elements has been avoided.

We now define an Up forest as a union of Up trees (with disjoint index sets).

**Proposition 4.** *The Up trees on a given index set $I$ with $m$ elements (in which all indices belong to at least one edge) are in bijective correspondence with the cyclic orderings of $[m] = \{1, \ldots, m\}$, or equivalently the orderings of $I$ starting with the smallest index. This number is clearly $(m-1)!$.*

*Proof.* It is fairly easy to see that Up trees are what is called 'recursive' - i.e. non-planar rooted trees with vertices labeled by distinct numbers, where the labels are strictly increasing as move in the direction of the arrows. It is a classical result that there are $(m-1)!$ of these on an index set of size $m$. One way to see it is to place the root at the bottom of the picture with the edges pointing up, and order the children of each node by increasing size toward the left (we can do this since the trees are non-planar, i.e. the children of each node are unordered): see the sample Up tree above. Now thicken all the edges into ribbons (which are kept flat to the plane with no twisting). Finally, starting at the root go along the outside edge of the ribbon graph in a clockwise direction writing down each index the first time it is reached. The result is an ordering of the $m$ indices starting with the smallest (in the case of the sample Up tree above we get $(i_1, i_4, i_6, i_5, i_3, i_2, i_7)$, and there are clearly $(m-1)!$ of these. It is easy to see that this procedure gives the required bijection. $\square$

**Corollary 2.** *The Up forests on an index set $I$ are in bijective correspondence with the unordered partitions of $I$ into cyclically ordered subsets.*

**Remark 9.** *The monomials corresponding to Up forests induced by unordered partitions of $[n] = \{1, 2, \ldots, n\}$) into cyclically ordered subsets form a basis for the algebra $\mathfrak{pb}^!_n$. Indeed, it is easy to see that Up forests are in bijective* *correspondence with the basis elements for $\mathfrak{pb}^!_n$ given in [Yuz], see also [Arnold] and [ShelYuz]. Also, the above excluded subgraphs correspond to the excluded monomials implied by the Gröbner basis given in [Yuz]. The fact that these Gröbner basis elements are quadratic allowed [ShelYuz] to conclude that $\mathfrak{pb}^!_n$ is Koszul.*

### 3.7.3 Up-Down Graphs and $\mathfrak{pvb}^!_n$

To define Up-Down graphs we first need the concept of an ordered 2-step partition (essentially due to [BarEnEtRa][14]). Namely given $n \in \mathbb{N}$ and $[n] := \{1, 2, \ldots, n\}$, first take an unordered partition of $[n]$ as $[n] = S_1 \sqcup \cdots \sqcup S_l$ where the sets $S_i$ are cyclically ordered (and let $m_i$ denote the minimal element of $S_i$). Second, take an unordered partition of the set $\mathcal{M} := \{m_i : i = 1, \ldots, l\}$ of minimal elements into distinct unordered subsets, $\mathcal{M} = M_1 \sqcup \cdots \sqcup M_k$.

Now suppose given such an ordered 2-step partition of $[n]$. First we form the Down forest on the index set $\mathcal{M}$ with the given partition. Next we form the Up tree on each of the (cyclically ordered) sets $S_i$. The resulting graph is called an **Up-Down forest** on the index set $[n]$. An Up-Down forest is uniquely determined by a particular ordered 2-step partition, and conversely.

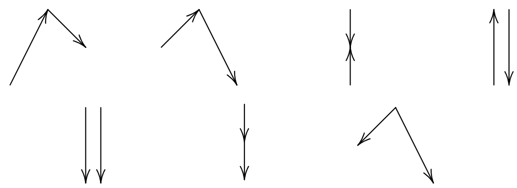
**Theorem 4.** *The monomials corresponding to Up-Down graphs on ordered 2-step partitions of $[n] = \{1, \ldots, n\}$ form a basis of $\mathfrak{pvb}^!_n$.*

**Remark 10.** *It is not hard to see that the Up-Down graphs on $[n]$ consist exactly of the red-black graphs corresponding to the monomials referred to in Proposition 4.5 of [BarEnEtRa].[15] These monomials are shown in that proposition to form a basis of a certain algebra $QA^0_n$ related to $\mathfrak{pvb}^!_n$: namely, after making a certain change of basis to $\mathfrak{pvb}^!_n = U(\mathfrak{qtr}_n)^!$, [BarEnEtRa] show that a certain filtration is defined on $\mathfrak{pvb}^!_n$. Then $QA^0_n$ is the quadratic approximation to the associated graded of $\mathfrak{pvb}^!_n$ with respect to that filtration. The given basis for $QA^0_n$ is then used to find the Hilbert series and to prove the Koszulness of $QA^0_n$, which in turn lead to the Hilbert series and Koszulness of $\mathfrak{pvb}^!_n$. It is interesting that the same collection of (Up-Down) graphs can be used to index a basis of $\mathfrak{pvb}^!_n$ itself and show directly that it is Koszul, as we shall see next.*

**Proposition 5.** *The algebra $\mathfrak{pvb}^!_n$ has a basis consisting of all monomials whose graph does not contain any of the following as subgraphs (again the relative heights of the endpoints indicate relative ordering of the indices, and the graphs are non-planar, so that the all edges (incoming or outgoing) incident to a particular vertex may be represented in any order without changing the graph):*

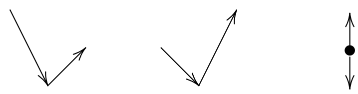
[14] See the proof of Corollary 4.6, (iii). Our *ordered* 2-step partitions differ from their '2-step partitions' in that our underlying sets $S_i$ are cyclically ordered and theirs are unordered.

[15] The Down and Up graphs correspond respectively to red and black graphs in the terminology used in the definition of 2-step partition immediately prior to Proposition 4.5 of [BarEnEtRa].



Note that the first row consists exactly of the excluded subgraphs for Up graphs. Thus if, in a graph corresponding to a basis monomial for $\mathfrak{pvb}^!_n$, we look only at the subgraph of upward arrows, we see that this subgraph must be an Up graph (and all Up graphs may arise).

The second row of excluded subgraphs features 'mixed' subgraphs, in that they each involve both an up arrow and a down arrow. It is clear that the non-excluded (= permitted) mixed subgraphs must be the following:



As is readily seen, the effect of these excluded and non-excluded mixed subgraphs is to ensure that different Up trees (which use only up arrows) which are connected to each other by down arrows are in fact only connected to each other by down arrows between their minimal elements.

Finally, the last row of excluded subgraphs involve only downward pointing arrows, and consist precisely of the excluded subgraphs for Down graphs. Thus a graph which excludes all the subgraphs listed in the proposition will be an Up-Down graph, and conversely. Thus Theorem 4 follows from Proposition 5.

*Proof of the Proposition.* To begin with we linearly order the generators $\{r_{ij} : 1 \leq i \neq j \leq n\}$ of $\mathfrak{pvb}^!_n$ using the numerical order of the indices, i.e. $r_{ij} > r_{kl} \iff (i > k)$ or $(i = k$ and $j > l)$. Then, given a wedge product of generators, we first order the generators in the product in increasing order, and then we linearly order such monomials first by length and then lexicographically (we also agree that 0 has length 0, so that $0 < u$ for all non-zero $u$). This ordering (which we refer to as the lexicographical ordering) is multiplicative in the sense that if $u, v, w$ are wedge products such that $u > v$ and $uw \neq 0$ then $uw > vw$.

We define a set $S^{(2)}$ of 'illegal' degree 2 monomials, consisting of those degree 2 monomials which can be expressed as linear combinations of 'smaller' monomials (with respect to the lexicographical ordering) using the defining relations of $\mathfrak{pvb}^!_n$. The set $S^{(2)}$ cannot be read off directly from the relations in the form (16), (17) and (18) as some of these have the same maximal terms. However one readily finds that those relations can be put in the following equivalent form (where $1 \leq i < j < k \leq n$):

$$r_{ik} \wedge r_{jk} = r_{ij} \wedge r_{jk} - r_{ji} \wedge r_{ik} \tag{25}$$
$$r_{kj} \wedge r_{ji} = r_{ji} \wedge r_{ik} - r_{ji} \wedge r_{jk} - r_{ji} \wedge r_{ki} \tag{26}$$
$$r_{ki} \wedge r_{kj} = r_{ki} \wedge r_{ij} - r_{ji} \wedge r_{ik} + r_{ji} \wedge r_{jk} + r_{ji} \wedge r_{ki} \tag{27}$$
$$r_{ik} \wedge r_{kj} = r_{ij} \wedge r_{jk} - r_{ij} \wedge r_{ik} \tag{28}$$
$$r_{jk} \wedge r_{ki} = r_{ji} \wedge r_{ik} - r_{ji} \wedge r_{jk} \tag{29}$$
$$r_{ij} \wedge r_{kj} = r_{ij} \wedge r_{jk} - r_{ij} \wedge r_{ik} - r_{ki} \wedge r_{ij} \tag{30}$$

as well as the relations (18). Each relation now has a distinct maximal term, and these have been collected on the LHS above. Thus $S^{(2)}$ consists of the union of the sets:

$$\{r_{jk} \wedge r_{ik},\ r_{kj} \wedge r_{ji},\ r_{kj} \wedge r_{ki} : 1 \leq i < j < k \leq n\} \tag{31}$$
$$\{r_{ik} \wedge r_{kj},\ r_{jk} \wedge r_{ki},\ r_{ij} \wedge r_{kj},\ 1 \leq i < j < k \leq n\} \tag{32}$$
$$\{r_{ij} \wedge r_{ji},\ r_{ij} \wedge r_{ij} : 1 \leq i \neq j \leq n\} \tag{33}$$

These monomials are readily seen to correspond with the excluded diagrams of the Proposition. The Proposition will be proved if we can show that the set $\overline{S}$ of monomials which do not contain any of the excluded 2-letter monomials $S^{(2)}$ (even after re-ordering of the generators forming the monomial) comprise a basis for $\mathfrak{pvb}^!_n$.

The proof of this fact is in the following two steps:

- show that the set $\overline{S}$ generates $\mathfrak{pvb}^!_n$; and
- show that $\overline{S}$ has the same number of elements in each degree as the basis for $\mathfrak{pvb}^!_n$ given in Theorem 3 (which implies that the elements of $\overline{S}$ are linearly independent, and hence form a basis).

The fact that $\overline{S}$ generates $\mathfrak{pvb}^!_n$ is easy, since if we have a monomial which contains (possibly after reordering its factors) an excluded 2-letter monomial, we can replace the monomial by a sum of terms in which the excluded 2-letter monomial is replaced by a smaller, legal 2-letter monomial. It is clear that all of these terms are strictly smaller than the original monomial with respect to the lexicographical ordering, because of the multiplicative property of that ordering. Hence, repeating if necessary, we must eventually reach a sum of terms none of which contains an excluded 2-letter submonomial, even after reordering of its factors - i.e. a sum of terms belonging to $\overline{S}$.

The fact that $\overline{S}$ has the same number of elements in each degree as the basis for $\mathfrak{pvb}^!_n$ given in Theorem 3 is also straightforward. Let us consider again the procedure described above for creating Up-Down graphs:

- First, take an unordered partition of $[n]$ into some number $l \leq n$ of cyclically ordered subsets (and form the unique Up graphs determined by the cyclically ordered subsets) - the number of ways of doing this is $s(n,l)$, where $s(-,-)$ denotes (unsigned) Stirling numbers of the first kind. It is easy to see that the resulting Up forests have $(n-l)$ arrows, so that the resulting monomials have degree $(n-l)$. We let $m_i$ denote the minimal element of cycle $C_i$ for $i-1,\ldots,l$.

- Second, take an unordered partition of $\mathcal{M} := \{m_i : i = 1,\ldots,l\}$ as $\mathcal{M} = M_1 \sqcup \cdots \sqcup M_k$, where the $M_i$ are unordered, and form the unique Down graph determined by this partition of $\mathcal{M}$. The number of ways of doing this is $S(l,k)$, where $S(-,-)$ denotes (unsigned) Stirling numbers of the second kind. It is easy to see that the resulting Down forests have $(l-k)$ arrows, so that the resulting monomials have degree $(l-k)$.

It is clear that the resulting Up-Down graph will have $(n-k) = (n-l)+(l-k)$ arrows, and hence will correspond to a degree $(n-k)$ monomial.

Thus if $\bar{S}^{n-k}$ denotes the monomials in $\bar{S}$ of degree $(n-k)$ we find:

$$\dim\, \bar{S}^{n-k} = \sum_{l=k}^{n} s(n,l)S(l,k) = L(n,k) = \dim\, A^{!(n-k)}$$

For the last equality we used Corollary 1, and for the second-last equality we used the so-called Lah-Stirling identity:

$$L = sS$$

where $L$, $s$ and $S$ are infinite-dimensional lower-triangular matrices whose $(n,k)$-th entries are, respectively, $L(n,k)$ (Lah numbers of Corollary 1), $s(n,k)$ and $S(n,k)$. See [Riordan].

This completes the proof. □

**Corollary 3.** *The algebra* $\mathfrak{pvb}_n^!$ *(and hence also* $\mathfrak{pvb}_n$*) is Koszul.*

*Proof.* The fact that the monomials $\overline{S}$ not containing any of the 2-letter monomials $S^{(2)}$ form a basis for $\mathfrak{pvb}_n^!$ means that the equations (25)-(30) and (18) (whose leading terms are the $S^{(2)}$) constitute a Gröbner basis for $\mathfrak{pvb}_n^!$ (as a skew-commutative algebra - see [Mikha]). This Gröbner basis is quadratic, and hence by a result of [Yuz] [16], $\mathfrak{pvb}_n^!$ is Koszul. □

**Remark 11.** $\mathfrak{pvb}_n^!$ *as a 'Product' of the families* $\mathfrak{pfb}_n^!$ *and* $\mathfrak{pb}_n^!$

Given the correspondence between Down forests and $\mathfrak{pfb}_n^!$, and between Up forests and $\mathfrak{pb}_n^!$, identified in Remarks 8 and 9, Theorem 4 suggests that the family of all $\mathfrak{pvb}_n^!$ (parametrized by $n$) may be some kind of 'product' of the families of the $\mathfrak{pfb}_n^!$ and $\mathfrak{pb}_n^!$. Indeed, one could express the Lah-Stirling identity above in the form:

$$\dim\, \mathfrak{pvb}_n^{!n-k} = L(n,k) = \sum_l s(n,l)S(l,k) = \sum_l \dim\, \mathfrak{pb}_n^{!(n-l)}\, \dim\, \mathfrak{pfb}_l^{!(l-k)}$$

[16] Theorem 6.16.

As pointed out in [BarEnEtRa], $\mathfrak{pb}_n$ may be viewed as a quotient of $\mathfrak{pvb}_n$ by $\mathfrak{pfb}_n$. However, this does not explain why one might be able to view $\mathfrak{pvb}_n^!$ as the kind of product of the families $\mathfrak{pb}_n^!$ and $\mathfrak{pfb}_n^!$ suggested by the Lah-Stirling identity.

### 3.8 Iterative Structure of $\mathfrak{pvb}_n$

In this subsection we will establish the following result:

**Theorem 5.** *For all $n \geq 3$, $\mathfrak{pvb}_n$ is a free $\mathfrak{pvb}_{n-1}$ module with respect to the module structure induced by the inclusion $\mathfrak{pvb}_{n-1} \hookrightarrow \mathfrak{pvb}_n$.*

This result in fact follows from the existence of the quadratic Gröbner basis for $\mathfrak{pvb}_n^!$ exhibited in Corollary 3, with little extra work other than to invoke a number of known results from the theory of Koszul algebras.

We first need to recall the definition of **Koszul module** (which we repeat from [Pol][17]). A graded module $M$ over a Koszul algebra $A$ is called Koszul if it has a linear free resolution as an $A$-module, that is a resolution:

$$\cdots \to A \otimes V_2 \to A \otimes V_1 \to A \otimes V_0 \to M \to 0 \tag{34}$$

where each $V_i$ is a vector space concentrated in degree $i$. Note that an algebra $A$ is Koszul iff $\mathbb{Q}$ (with the $A$-module structure induced by the augmentation map $A \to \mathbb{Q}$) has a linear free resolution.

We will use the following proposition, which is also taken from [Pol][18]:

**Proposition 6.** *Let $f : A \to B$ be a homomorphism of Koszul algebras. Then $B$ is a Koszul $A$-module iff $A^!$ is a free $B^!$-module with respect to the module structure induced by the (quadratic) dual homomorphism $f^! : B^! \to A^!$.*

*Proof of Theorem 5.* We set $A = \mathfrak{pvb}_n^!$ and $B = \mathfrak{pvb}_{n-1}^!$ and take the map $f$ to be the natural projection $\Pi : \mathfrak{pvb}_n^! \twoheadrightarrow \mathfrak{pvb}_{n-1}^!$ which sends $r_{i,n}$ and $r_{n,i}$ to 0, and $r_{i,j}$ to itself, for all $i, j < n$.

The map $\Pi$ is induced from a similarly defined map on the tensor algebra with the same generators, which descends to a quotient map $\mathfrak{pvb}_n^! \to \mathfrak{pvb}_{n-1}^!$ (because the relations never mix monomials with and without the index $n$). The (quadratic) dual map is induced by the (linear) dual map on dual generators, which also descends to a quotient map $\mathfrak{pvb}_{n-1} \to \mathfrak{pvb}_n$ (for the same reason). Hence the quadratic dual map is just the inclusion $\mathfrak{pvb}_{n-1} \hookrightarrow \mathfrak{pvb}_n$ (i.e. killing dual generators which contain the index $n$ is dual to including the generators of $\mathfrak{pvb}_{n-1}$ into $\mathfrak{pvb}_n$ on the first $(n-1)$ indices).

Now the result follows from the fact that the projection $\Pi$ makes $\mathfrak{pvb}_{n-1}^!$ a Koszul $\mathfrak{pvb}_n^!$-module, which we prove in the next Proposition. $\square$

[17] See Definition 2(M) page 19 and the comments following it, and the 'Definition' which appears on page 9.

[18] See Corollary 2.5.9, page 35.

**Proposition 7.** *The map* $\Pi : \mathfrak{pvb}^!_n \twoheadrightarrow \mathfrak{pvb}^!_{n-1}$ *makes* $\mathfrak{pvb}^!_{n-1}$ *a Koszul* $\mathfrak{pvb}^!_n$*-module.*

*Proof.* The idea of the proof is to reduce the problem, by way of a spectral sequence, to the statement that $gr(\mathfrak{pvb}^!_{n-1})$ is a Koszul $gr(\mathfrak{pvb}^!_n)$ module (where $gr$ is determined with respect to a suitable filtration), and then prove this statement.

The existence of the linear free resolution (34) for $A = \mathfrak{pvb}^!_n$ and $M = \mathfrak{pvb}^!_{n-1}$ is equivalent to the statement that $Tor^{\mathfrak{pvb}^!_n}_{ij}(\mathbb{Q}, \mathfrak{pvb}^!_{n-1}) = 0$ for $i \neq j$. We compute this Tor by applying the functor $- \otimes_{\mathfrak{pvb}^!_n} \mathfrak{pvb}^!_{n-1}$ to the bar resolution for $\mathfrak{pvb}^!_n$, and thus obtain a complex whose homological degree-$i$ term is:[19]

$$\cdots \to (\mathfrak{pvb}^!_n)^{\otimes i+1} \otimes_{\mathfrak{pvb}^!_n} \mathfrak{pvb}^!_{n-1} \to \ldots$$

which is to say

$$\cdots \to (\mathfrak{pvb}^!_n)^{\otimes i} \otimes \mathfrak{pvb}^!_{n-1} \to \ldots$$

The total order on (anti-symmetric) monomials introduced in the previous section induces a filtration on the above complex, and this will allow us to compute the Tor by using a spectral sequence. To explain this induced filtration, we need to re-phrase the total order on anti-symmetric monomials in an equivalent way, which extends more reasonably to the bar complex - more specifically, it extends in such a way that the associated graded of the bar complex is the bar complex of $gr(\mathfrak{pvb}^!_{n-1})$ as a $gr(\mathfrak{pvb}^!_n)$ module.

Recall that, given a total order on variables $x_1 < \cdots < x_N$, we get a total order on commutative monomials by positing that, for $x^a = x_1^{a_1} \ldots x_N^{a_N}$ and $x^b = x_1^{b_1} \ldots x_N^{b_N}$, we have $x^a < x^b$ iff there is a $k$, $1 \leq k \leq N$, such that $a_1 = b_1, \ldots, a_{k-1} = b_{k-1}$ but $a_k < b_k$. Then we recover the total order on anti-symmetric monomials given in the previous section by (a) ordering monomials by increasing length (with rational numbers including 0 considered to have length 0), and (b) given two non-zero wedge products of generators $u$ and $v$, $u < v$ whenever the same is true for the corresponding commutative products of generators.

We get a filtration $\mathcal{F}$ on skew-symmetric monomials by defining, for each wedge product $u$, $\mathcal{F}_u := \mathbb{Q}\{\text{wedge products } v \text{ s.t. } v \leq u\}$. This filtration induces a filtration (also denoted $\mathcal{F}$) on the bar complex by setting, for each *commutative* product $u$:

$$\mathcal{F}_u((\mathfrak{pvb}^!_n)^{\otimes i} \otimes \mathfrak{pvb}^!_{n-1}) := \sum_{v_1 \ldots v_{i+1} \leq u} \mathcal{F}_{v_1}(\mathfrak{pvb}^!_n) \otimes \cdots \otimes \mathcal{F}_{v_i}(\mathfrak{pvb}^!_n) \otimes \mathcal{F}_{v_{i+1}}(\mathfrak{pvb}^!_{n-1})$$

where the sum is over *wedge* products $v_l$ such that the *commutative* product over the underlying generators is $\leq u$.

One readily checks that the complex respects the filtration, i.e. $d\mathcal{F}_u \subseteq \mathcal{F}_u$. Indeed, for any tensor product $z$, $dz$ is just a sum of terms obtained

[19] All tensors are over $\mathbb{Q}$, except where specifically indicated otherwise, as in $\otimes_{\mathfrak{pvb}^!_n}$.

by multiplying neighbouring tensor components, and whenever these terms are non-zero they have the same underlying commutative products as $z$. If any of the resulting tensor components is an excluded monomial (in the sense of Proposition 5), it can be replaced by a sum of terms that are 'smaller' with respect to the total order on anti-symmetric monomials.

One might be tempted to simply obtain a filtration on the bar complex by taking the above sum to be over wedge products $v_l$ such that their wedge product (rather than their commutative product) is $\leq u$. The difficulty one runs into may be illustrated as follows. Recall the relation $r_{ik}\wedge r_{jk}-r_{ij}\wedge r_{jk}+r_{ji}\wedge r_{ik}=0$, where we suppose $i<j<k$, so that the left-most term $r_{ik}\wedge r_{jk}$ is maximal. If we tensor with $r_{ik}$, the left most term becomes $r_{ik}\wedge r_{jk}\otimes r_{ik}$ with total wedge product 0 (and similarly with the third term). After passing to the associated graded bar complex, we would get a relation $r_{ij}\wedge r_{jk}\otimes r_{ik}=0$ (from the middle term). However, neither $r_{ij}\wedge r_{jk}$ nor $r_{ik}$ is zero in $gr(\mathfrak{pvb}^!_n)$. Hence the associated graded of the bar complex would *not* be the bar complex of $gr(\mathfrak{pvb}^!_{n-1})$ as a $gr(\mathfrak{pvb}^!_n)$ module.

However it is not hard to check that, with the induced filtration as described, the associated graded of the bar complex is in fact the bar complex of $gr(\mathfrak{pvb}^!_{n-1})$ as a $gr(\mathfrak{pvb}^!_n)$ module. Indeed, relations in the filtered complex are just tensor multiples of the relations in $\mathfrak{pvb}^!_n$ (or $\mathfrak{pvb}^!_{n-1}$). Moreover, the ordering of commutative monomials is strictly multiplicative, i.e. if $u<v$ then $uw<vw$ for all non-zero monomials $u,v,w$. Hence taking tensor multiples of the relations preserves the maximal terms in those relations. As a result, after passing to the associated graded complex, we just get the bar complex of $gr(\mathfrak{pvb}^!_{n-1})$ as a $gr(\mathfrak{pvb}^!_n)$ module (or equivalently the bar complex computing $Tor^{gr(\mathfrak{pvb}^!_n)}_{ij}(\mathbb{Q},gr(\mathfrak{pvb}^!_{n-1}))$.

We therefore get a spectral sequence:

$$E_1=Tor^{gr(\mathfrak{pvb}^!_n)}_{ij}(\mathbb{Q},gr(\mathfrak{pvb}^!_{n-1}))\Longrightarrow Tor^{\mathfrak{pvb}^!_n}_{ij}(\mathbb{Q},\mathfrak{pvb}^!_{n-1})$$

Thus the result will follow if we can prove that $Tor^{gr(\mathfrak{pvb}^!_n)}_{ij}(\mathbb{Q},gr(\mathfrak{pvb}^!_{n-1}))=0$ for $i\neq j$, which is the content of the following lemma. □

**Lemma 5.** *$gr(\mathfrak{pvb}^!_{n-1})$ is a Koszul $gr(\mathfrak{pvb}^!_n)$-module, i.e. $Tor^{gr(\mathfrak{pvb}^!_n)}_{ij}(\mathbb{Q},gr(\mathfrak{pvb}^!_{n-1}))=0$ for $i\neq j$.*

*Proof.* Essentially by construction, the map $\Pi:\mathfrak{pvb}^!_n\twoheadrightarrow\mathfrak{pvb}^!_{n-1}$ constructed above has kernel the ideal $K_n$ in $\mathfrak{pvb}^!_n$ generated by the elements $r_{i,n}$ and $r_{n,i}$. It is clear that $\Pi$ respects the filtration, and that the associated graded map, $gr\Pi:gr(\mathfrak{pvb}^!_n)\twoheadrightarrow gr(\mathfrak{pvb}^!_{n-1})$ has kernel the ideal $k_n$ in $gr(\mathfrak{pvb}^!_n)$ generated by the same elements.

Here is where we use the fact that we have a quadratic Gröbner basis for $\mathfrak{pvb}^!_n$. The existence of this basis implies that $gr(\mathfrak{pvb}^!_n)$ is a quotient of an exterior algebra by (relations generated by) *quadratic* relations. But then, since the filtration we use to determine the associated graded is induced from a total ordering of the monomials in $\mathfrak{pvb}_n^!$, it follows that $gr(\mathfrak{pvb}_n^!)$ is a quotient of an exterior algebra by *monomial* quadratic relations.

Finally, the following Theorem from [Pol][20] implies that $gr(\mathfrak{pvb}_{n-1}^!) = gr(\mathfrak{pvb}_n^!)/k_n$ is a Koszul $gr(\mathfrak{pvb}_n^!)$-module. □

**Theorem 6.** *Let SP be a skew-polynomial algebra, i.e. a quadratic algebra with generators $x_1, \ldots, x_N$ and relations*

$$x_i x_j = q_{ij} x_j x_i, \quad i < j$$

*where the $q_{ij}$ are non-zero constants. If $A$ is a quotient algebra of SP by monomial quadratic relations, and $I$ is an ideal of $A$ generated by some subset of the $\{x_i\}$, then $A/I$ is a Koszul $A$-module.*

# 4 Final Remarks

## 4.1 Other Groups

One could seek to apply the PVH Criterion to determine whether other groups are quadratic. One group that comes to mind is the pure cactus group $\Gamma$, as developed for instance in [EHKR]. As a prerequisite, one would need to have a presentation for the pure cactus group, and to show that the quadratic approximation to the associated graded of $\mathbb{Q}\Gamma$ with respect to the filtration by powers of the augmentation ideal (i.e. the universal enveloping algebra of the holonomy Lie algebra of $\Gamma$) is Koszul (at least up to homological degree 2).

## 4.2 Generalizing the PVH Criterion

As mentioned in the Introduction, the PVH Criterion arguably lives naturally in a broader context than we have explored here, such as perhaps augmented algebras over an operad (or the related 'circuit algebras' of [BN-WKO]).

In a different direction, one could try to generalize the criterion to deal with filtrations of an algebra by powers of an ideal other than an augmentation ideal. A particular case of this deals with groups that exhibit a 'fibering'. For instance the virtual braid group $vB_n$ fits into an exact sequence:

$$1 \to PvB_n \to vB_n \to S_n \to 1$$

where $S_n$ is the symmetric group. (Similar sequences exist for the braid group and the cactus group.) In such cases it is more interesting to consider the ideal corresponding to the kernel of the induced homomorphism $\mathbb{Q}vB_n \to \mathbb{Q}S_n$, rather than the augmentation ideal of $\mathbb{Q}vB_n$. The extension of the PVH Criterion to cover these particular ideals should not be too difficult, but dealing with more general ideals could be interesting.

[20] See Theorem 4.8.1 page 92, and its proof. The theorem generalizes the result of [Fröberg1].

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